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NEW COMPOSITE RELATIONS CHARACTERIZATION OF STAR-LIKE FINITE SEMIGROUPS

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ABSTRACT. The study establishes new star-like classical finite semigroups, provides a cohesive explanation, appraises some combinatorial composite relations on $\alpha\omega_n^*$, and proves some combinatorial relations of star-like $P\omega_n^*$, $T\omega_n^*$, and $O^{cd}P\omega_n^*$ star-like-partial ordered connected transformation semigroups. Let $\alpha\omega_n^*$ be a star-like transformation semigroup with a star-like diskpoint m^*, diagonal difference operator $\nabla^*_{b_{(n,m^*)}}$, and vertical difference operator $\Delta^*_{(n,r^*)}$. The study shows that star-like sequences converge uniformly and are reducible if $|v^*(\alpha^*_{(i,j)})| = F(n,v^*)$. Also, for a star-like diskpoint m^*, $F(n,m^*) = (n-1)^2 + (m+1)(n-2)$ and $|\Delta^*_{(n,r_3^*)}| = \lambda_i^{2*} + (n-2)$ such that $|c^+(\alpha^*)| \leq |c^-(\alpha^*)|$. The new combinatorial composite relations were generated from the existing ones by composition of star-like mapping, which was applied to star-like $\alpha\omega_n^*$ finite semigroups with emphasis on combinatorial triangular arrays.

1. INTRODUCTION AND BACKGROUND OF THE STUDY

A star-like partial relation (transformation) $\alpha^*_{(i,j)} : A^* \longrightarrow B^*$ is a rule $f : \alpha^*_{(i,j)} \longrightarrow B^*$ for some $\alpha^*_{(i,j)} \neq \emptyset, \alpha^*_{(i,j)} \subseteq A^*$. If A^* is a star-like set and $P\omega_n^*(A^*)$ denotes the set of all star-like partial transformations whose domain and range are subsets of A^*, then the transformation with domain $\alpha^* \cdot \beta^* \in P\omega_n^*(A^*)$ is composed of star-like transformations with

$$B^* = (range(\alpha^*_{(i,j)}) \cap domain(\beta^*_{(i,j)}))\alpha^{-1*}_{(i,j)}$$

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for every $a^* \in B^*$

$$a^*(\alpha^*_{(i,j)} \cdot \beta^*_{(i,j)}) = (a\alpha^*_{(i,j)})\beta^*_{(i,j)}.$$

Let $\alpha\omega_n^*$ be a star-like classical semigroup and assume (for simplicity) that $\alpha^*, \beta^* \in \alpha\omega_n^*$ are such that $\alpha^*\beta^{*2} = \beta^*$. If some star-like relations f^* and g^* were defined on $\alpha\omega_n^*$ by $f^*(P\omega_n^*) = \alpha^* P\omega_n^*$ and $g^*(P\omega_n^*) = \beta^* P\omega_n^*$ such that set $\beta^*\alpha\omega_n^* \cup \{\beta^*\}$ (where $\beta^*\alpha\omega_n^* = \{\beta^* P\omega_n^* \mid P\omega_n^* \subseteq \alpha\omega_n^*\}$) is considered. Then restriction of g^* to $\beta^*\alpha\omega_n^* \cup \{\beta^*\}$ and f^* to the range of g^* gives

$$\begin{aligned} g^* &: \beta^*\alpha\omega_n^* \cup \{\beta^*\} \to \beta^{*2}\alpha\omega_n^* \cup \{\beta^{*2}\}, \\ f^* &: \beta^{*2}\alpha\omega_n^* \cup \{\beta^{*2}\} \to \alpha^*\beta^{*2}\alpha\omega_n^* \cup \{\alpha^*\beta^{*2}\} = \beta^*\alpha\omega_n^* \cup \{\beta^*\} \end{aligned}$$

That is

$$\begin{aligned} g^* &: \beta^*\alpha\omega_n^* \cup \{\beta^*\} \to \beta^*\alpha\omega_n^* \cup \{\beta^*\}, \\ f^* &: \beta^*\alpha\omega_n^* \cap \{\beta^*\} \to \beta^*\alpha\omega_n^* \cap \{\beta^*\}. \end{aligned}$$

But as $\beta^*\alpha\omega_n^* \cup \{\beta^*\}$ is finite and $f^* \circ g^*$ is the star-like combinatorial identity on $\beta^*\alpha\omega_n^* \cup \{\beta^*\}$, $g^* \circ f^*$ is also the star-like combinatorial identity on $\beta^*\alpha\omega_n^* \cup \{\beta^*\}$. Therefore, $\beta^*\alpha^*\beta^* = g^*(f^*(\beta^*)) = \beta^*$.

Suppose $N_i = \{i,\ i+1,\ i+2,\ \cdots\}(i = \{0,1,2,\cdots\}) \in Z_n$ is a finite chain under the natural ordering and $P\omega_n^* \subseteq \alpha\omega_n^*$ is the star-like partial transformation on Z_n under usual star-like composition, then star-like transformation $\alpha^* : D(\alpha^*) \subseteq Z_n \to I(\alpha^*) \subseteq Z_n$ is star-like full if $I(\alpha^*) = Z_n$, denoted by $T\omega_n^*$; star-like partial if $I(\alpha^*) \subseteq D(\alpha^*)$, denoted by $P\omega_n^*$. The pivot is denoted and defined as $V^+(\alpha^*) = |\frac{n \cdot r^+(\alpha^*)}{c^+(\alpha^*)+c^-(\alpha^*)}|$. The relapse of α^* is denoted and defined as $c^-(\alpha^*) = |n - c^+(\alpha^*)|$. The collapse of α^* is denoted by $c^+(\alpha^*)$ and defined as $c^+(\alpha^*) = |\bigcup_{i=1}^n y_i\alpha^{-1*} : |y_i\alpha^{-1*}| \geq 2|$.

Some other combinatorial relations have been defined in [13, 14], and more could still be defined, but this research work is restricted to these only. Just as two real integers can be combined to make other real integers using fundamental arithmetic operations, two combinatorial functions can be combined to form new functions where the output of one combinatorial function becomes the input of another. The domain of the union of star-like relations f^* and g^* is made up of all real integers that are shared by the domains of $f^* \cup g^*$. For the benefit of future research, the newly discovered combinatorial composite relations were exploited to generate open issues and some valuable conclusions on classical finite transformation semigroups. The number of idempotents in P_n was obtained by [5], and for P_n, the number of nilpotents is given by

$$|N(P_n)| = (n+1)^{n-1}$$

which is deduced from [10]. The collapsible element for $|t\alpha^{-1}| = 2$ and $|t\alpha^{-1}| = 3$ for all $n \geq 2$ $(n \in \mathbb{N})$ in T_n was studied by [1], while [11] studied the collapsible element for $|t\alpha^{-1}| = 2$ and $|t\alpha^{-1}| = 3$ for all $n \geq 2$ $(n \in \mathbb{N})$ in P_n. Matrix notations were used to present the sequential elements of these star-like classical finite transformations, as used by [12].

To define the star-like pivot, depth, collapse, and relapse of a star-like transformation in a combinatorial composite relation, the base set Z_n must be completely ordered. It is necessary to carry out the reverse process, decomposing a complicated combinatorial composite relation into one or more simple relations, whereby the order in which the composition occurs is fixed. [4] obtained several important results in the semigroup P_n, and [10] examined combinatorial relations of certain subsemigroups of P_n and discovered equivalent results. However, no analogous results for the star-like semigroup $\alpha\omega_n^*$ or any of its subsemigroups have been found to match all of these developments concerning combinatorial relations of P_n and some of its subsemigroups.

This research work focuses on $P\omega_n^*$, the star-like partial semigroup; $T\omega_n^*$, the star-like full semigroup; $O^{cd}P\omega_n^*$, the star-like order connected transformation semigroups on Z_n, which was inspired by the study of [2]. The algebraic structure of the new classical star-like $T\omega_n$ semigroup was established; obtain recurrence relations such that the cardinality of $T\omega_n^*$ is similar to the one of CP_n using some combinatorial parameters; then give an overview of our results and make concluding remarks. The following is a list of papers and books that is by no means exhaustive: [3, 4, 7, 8]. For basic and standard concepts in the theory of transformation semigroups, refer to one of the following: [3, 5, 6, 9, 10].

2. PRELIMINARY LEMMAS AND RESULTS

Let $\Delta^*_{(i,j)}$ be the star-like operator order for all vertical elements of $\alpha^* \in T\omega_n^*$, and $\nabla^*_{(i,j)}$ be the star-like operator order for all diagonal elements of $\alpha^* \in T\omega_n^*$ with $Z_n = \{1, 2, 3, \ldots\}$ as a star-like distinct finite n-element set. The semigroups $P\omega_n^*$, $T\omega_n^*$, $D^2I\omega_n^*$, $O^{cd}\omega_n^*$, and $B\omega_n^*$ are star-like classical finite transformations such that

$$|\alpha^* w_i - w_{i+1}| \leq |\alpha^* w_{i+1} - w_i| \tag{1}$$

for each $w_i \in D(\alpha^*)$ and $\alpha^* w_i \in I(\alpha^*)$, where $\mathbb{N}_i \cup \emptyset$; $N_i = \{i,\ i+1,\ i+2,\ \cdots\}(i = \{0,1,2,\cdots\})$. The star-like transformation on Z_n is given as an array of the form:

$$\alpha_n^* = \begin{pmatrix} w_1 & w_2 & w_3 & \ldots & w_n \\ \alpha^* w_1 & \alpha^* w_2 & \alpha^* w_3 & \ldots & \alpha^* w_n \end{pmatrix}. \tag{2}$$

Then, equation (2) is a star-like function $\alpha^* : A^* \longrightarrow B^*$, where $A^* = \{l_1, l_2, l_3, \ldots l_n\}$ is a subset of B^*. Consider some elements of α^*, such as:

$$\alpha^* = \begin{pmatrix} l_1 & l_2 & l_3 & \ldots & l_n \\ \alpha^* l_1 & \alpha^* l_2 & \alpha^* l_3 & \ldots & \alpha^* l_n \end{pmatrix}. \tag{3}$$

The set of all star-like transformations of $\alpha\omega_n^*$ on Z_n would be denoted as α_i^*. Therefore, the elements of $\alpha\omega_n^*$ in equation (2) have the form:

$$\alpha_{(i,j)}^* = \begin{pmatrix} 1 & 2 & 3 & \ldots & n \\ \alpha^* w_1 & \alpha^* w_2 & \alpha^* w_3 & \ldots & \alpha^* w_n \end{pmatrix}. \tag{4}$$

The results shows that any transformation $\alpha_{(i,j)}^*$ in $\alpha\omega_n^*$ defined in equation (1) is a special case of $\alpha\omega_n^*$ which is a star-like mapping from a set to itself such that the star-like composition of any two or more star-like mappings of the same set gives the same transformations of the set.

Definition 1: Suppose that α^* and β^* are star-like transformations, and $f^* : \alpha^* \longrightarrow \beta^*$ is a star-like rule. Then f^* is a star-like composite invertible if there exists a star-like composite rule. $g^* : \beta^* \longrightarrow \alpha^*$ such that $g^* \circ f^* = e_{\alpha_i^*}{}^*$ and $f^* \circ g^* = e_{\beta_i^*}{}^*$.

Definition 2: Star-like Mapping ($\longrightarrow^*$): Let Z_n and Y_n be non-empty star-like sets. A relation $\alpha_{(i,j)}^*$ from Y_n into Z_n is called a mapping from Z_n into Y_n if

(i) $D(\alpha_{(i,j)}^*) = Z_n$

(ii) for all $z, n, k', l' \in \alpha_{(i,j)}^* \Longrightarrow x = k'$, $n = l'$.

Definition 3: For any star-like combinatorial composite rule f^* and g^*, the domain of $f^* \circ g^*$ star-like combinatorial composition of f^* and g^* is obtained if

$$\begin{aligned} (f^* \circ g^*)(u) &= f^*(g^*(u)) \ and \\ D(f^* \circ g^*) &= \{u \in D(g^*) | g^*(u) \in D(f^*)\}. \end{aligned}$$

Definition 4: Suppose that α^* and β^* are star-like transformations, and $f^* : \alpha^* \longrightarrow \beta^*$ assigns a unique element $q = f^*(u) \in \beta^*$ to every $q \in \alpha^*$. Then the order of two star-like combinatorial composite rules $f^* : \alpha^* \longrightarrow \beta^*$ and $g^* : \alpha^* \longrightarrow \beta^*$ is equal to $(\alpha^* = \beta^*)$ if and only if $f^*(u) = g^*(u)$ for all $u_i \in \beta^*$.

Definition 5: A star-like combinatorial composite rule: $f^* : \alpha^* \longrightarrow \beta^*$ is a star-like one-one (injection) if whenever $u, q \in \alpha^*$ there are such that $f^*(u) = f^*(q)$ then $u = q$.

Definition 6: A star-like combinatorial composite rule $f^* : \alpha^* \longrightarrow \beta^*$ is onto (star-like surjective) if for all $q \in \beta^*$, $u \in \alpha^*$ exists such that $f^*(u) = q$. Then $g^* \circ f^* : \alpha^* \longrightarrow \beta^*$ can be defined as $(g^* \circ f^*)(u) = g^*(f^*(u))$.

Definition 7: For any given star-like transformation $\alpha_i^* \in P\omega_n^*$, there exists a unique identity star-like rule $e^*_{\alpha_i^*} : \alpha_i^* \longrightarrow \alpha_i^*$ defined by $e_{\alpha_i^*}{}^*(u) = u$ for all $u \in \alpha_i^*$.

Lemma 1: For all positive integers n, c, and d in $N_i = \{i,\ i+1,\ i+2,\ \cdots\}(i = \{0,1,2,\cdots\})$ we have

$$\sum_{k=d}^{n} \left(\begin{array}{c} k \\ d \end{array} \right) \left(\begin{array}{c} n+c-k \\ c \end{array} \right) = \left(\begin{array}{c} n+c+1 \\ c+d+1 \end{array} \right).$$

Lemma 2: Let (u, v) denote numbers of a sequence path from (e_0, e_0) to (u, v) with u-row and v-column of a star-like triangular array obtained from the sequence of $\alpha\omega_n^*$, then:

$$T(u,\ e_0) = T(e_0,\ v) = 1$$
$$T(u,\ v) = T(u,\ v-1) + T(u-1,\ v),$$

such that

$$\left(\begin{array}{c} u+v \\ u \end{array} \right) + \left(\begin{array}{c} u+v \\ v \end{array} \right) = \frac{(u+v)!}{e_0!}$$

for all $u,\ v\ \in \mathbb{N} \cup\ 0\ \in \mathbb{R}$.

Lemma 3: Suppose star-like combinatorial compositions are associative, and if α^*, β^*, γ^*, and ξ^* are star-like transformations in $P\omega_n^*$ where $g^* : \beta^* \longrightarrow \gamma^*$ and $h^* : \gamma^* \longrightarrow \xi^*$ are star-like rules, then $h^* \circ (g^* \circ f^*) = (h^* \circ g^*) \circ f^*$.

Lemma 4: Suppose that $f^*:\alpha^* \longrightarrow \beta^*$ is a star-like composite rule. There is a $g^*:\beta^* \longrightarrow \alpha^*$ such that $g^{-1*}:\alpha^* \longrightarrow \beta^*$ implies $g^* \circ g^{-1*} = eg^*$. Then $g^{-1*}e_{\beta_i^*}{}^* = eg^*$.

Proof. If $g^{-1*} \circ f^* = e_{\alpha_i^*}{}^*$ and $f^* \circ g^{-1*} = e_{\beta_i^*}{}^*$, then definition (1) shows that

$$\begin{aligned} g^* = e_{\alpha_i^*}{}^* \circ g^* &= (g^{-1*} \circ f^*) \circ g^* = g^{-1*} \circ (f^* \circ g^*) \\ &= g^{-1*}(f^* \circ g^*) \\ &= g^{-1*} e_{\beta_i^*}{}^* \qquad\qquad = g^*. \end{aligned}$$

□

Proposition 5: Let $\alpha^* \in T\omega_n^*$ be a star-like transformation under the composition of star-like bijective mapping, then

$$3!|T\omega_n^*| = \begin{pmatrix} 13n^4 \\ 13r^{4*}-1 \end{pmatrix} - \begin{pmatrix} 5n^2 \\ 5r^{2*}-1 \end{pmatrix} \begin{pmatrix} 23n-76 \\ 23r^*-75 \end{pmatrix} - 2 \begin{pmatrix} 259n-5!+3 \\ 259r^*-5!+2 \end{pmatrix}$$

such that $n = r^*$; $r^+(\alpha^*) \geq 1$ where,

$$|r^+(\alpha^*)| \subseteq |T\omega_n^*| \subseteq |\alpha\omega_n^*|$$

for all $n \in \mathbb{N}_i \cup \emptyset$; $N_i(i = \{0,1,2,\cdots\})$.

Proof. Suppose $\alpha^* \in T\omega_n^*$ is a star-like transformation as in equation (1) such that if $\alpha^* \in \alpha\omega_n^*$ where $r^+(\alpha^*) \leq 1$ and $r^+(\alpha^*) \geq 1$, there exist star-like sequences w_n with $\Delta^*_{(i,j)}$ star-like order of vertical elements if $w_n = |T\omega_n^*|$. The sequence

$$w_n = \Delta_{w_1}\begin{pmatrix} n \\ 1 \end{pmatrix} + \Delta^2_{w_2}\begin{pmatrix} n \\ 2 \end{pmatrix} + \Delta^3_{w_3}\begin{pmatrix} n \\ 3 \end{pmatrix} + \cdots + \Delta^{k+1}_{w_n}\begin{pmatrix} n \\ k \end{pmatrix} \tag{5}$$

possess a unique positive integer difference. 52 at $\Delta^{4*}_{(n,r^+(\alpha^*))}$ for all $i \geq n \geq 1$; $N_1 = \{1,2,3,\cdots\}$.
Then $r^+(\alpha^*) \leq 1$ and $r^+(\alpha^*) \geq 1$ generate a system of equations:

$$\begin{array}{ccccccccccc} w_0 & + & w_1 & + & w_2 & + & w_3 & + & w_4 & = & 1 \\ 16w_0 & + & 8w_1 & + & 4w_2 & + & 2w_3 & + & w_4 & = & 3 \\ 81w_0 & + & 27w_1 & + & 9w_2 & + & 3w_3 & + & w_4 & = & 10 \\ 256w_0 & + & 64w_1 & + & 16w_2 & + & 4w_3 & + & w_4 & = & 37 \\ 625w_0 & + & 125w_1 & + & 25w_2 & + & 5w_3 & + & w_4 & = & 151. \end{array} \tag{6}$$

Since $\alpha^* \in T\omega_n^*$ is bijective, the above system in equation (6) may be re-written as $AW = B$, where

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & \vdots & 1 \\ 16 & 8 & 4 & 2 & \vdots & 1 \\ 81 & 27 & 9 & 3 & \vdots & 1 \\ 256 & 64 & 16 & 4 & \vdots & 1 \\ 625 & 125 & 25 & 5 & \vdots & 1 \end{pmatrix}, W = \begin{pmatrix} w_0 \\ w_1 \\ w_2 \\ w_3 \\ w_4 \end{pmatrix}, B = \begin{pmatrix} 1 \\ 3 \\ 10 \\ 37 \\ 151 \end{pmatrix}.$$

Maple 18 was used to obtain

$$w_0 = \frac{13}{6}, w_1 = \frac{-115}{6}, w_2 = \frac{190}{3}, w_3 = \frac{-518}{6}, w_4 = 41 \tag{7}$$

such that the value of w_n in equation (7) was carefully substituted into equation (6) to obtain

$$|T\omega_n^*| = \begin{pmatrix} 13n^4 \\ 13r^{4*} - 1 \end{pmatrix} - \begin{pmatrix} 5n^2 \\ 5r^{2*} - 1 \end{pmatrix} \begin{pmatrix} 23n - 76 \\ 23r^* - 75 \end{pmatrix} - 2 \begin{pmatrix} 259n - 5! + 3 \\ 259r^* - 5! + 2 \end{pmatrix}$$

which gives the required star-like recursive relation of $T\omega_n^*$ for all $r^+(\alpha^*) \leq 1$ and $r^+(\alpha^*) \geq 1$. $\square$

Theorem 6: If $\alpha^* \in T\omega_n^*$ is a star-like transformation and $T\omega_n^* \subseteq \alpha\omega_n^*$ is a star-like classical finite semigroup under the composition of mappings, then $|T\omega_n^*|$ is reducible if

$$F(n, r^*) = 2!\nabla^*_{(n, r^+(\alpha^*))} = -2!\nabla^*_{(n, r^+(\alpha^*))}; n \geq 2.$$

Proof. Suppose $\alpha^* \in T\omega_n^*$, then $F(n, r^*)$ has a unique negative integer difference (-3) at $\nabla^{3*}_{(n,r^*)}$ for $N_i (i = \{0, 1, 2, \cdots\})$ where $N_2 = \{2, 3, 4, \cdots\}$. Then

$$F(n, r^*) = \begin{pmatrix} 3! \\ 5 \end{pmatrix} - \begin{pmatrix} n-1 \\ r^*+1 \end{pmatrix}^3 + 11 \begin{pmatrix} n-1 \\ r^*+1 \end{pmatrix}^2 - 14 \begin{pmatrix} n-1 \\ r^*+1 \end{pmatrix}$$

whenever $r^* = n - 1$ for all $r^* \geq 2 : N_2 = \{2, 3, 4, \cdots\}$. But in the case of a large value of n, if $F(n, r^*) = |r^+(\alpha^*)|$, $\nabla^*_{(n,r^*)}$ has both positive and negative constant finite elements in the third order. Now, changing the sign operator to $|r^+(\alpha^*)|$ yields

$$2!\nabla^*_{(n, r^+(\alpha^*))} = \begin{pmatrix} n-1 \\ r^*+1 \end{pmatrix}^3 - 11 \begin{pmatrix} n-1 \\ r^*+1 \end{pmatrix}^2 + 14 \begin{pmatrix} n-1 \\ r^*+1 \end{pmatrix} - \begin{pmatrix} 3! \\ 5 \end{pmatrix}$$

as the reverse order for the element of $|r^+(\alpha^*)|$ for all $n \geq 2$. Thus

$$|r^+(\alpha^*)| = \Delta^*_{w_1} \begin{pmatrix} n \\ 1 \end{pmatrix} + \Delta^{2*}_{w_2} \begin{pmatrix} n \\ 2 \end{pmatrix} - \Delta^{3*}_{w_3} \begin{pmatrix} n \\ 3 \end{pmatrix}. \tag{8}$$

It follows in equation (6) that $T\omega_n^*$ is reducible if

$$|r^+(\alpha^*)| = \begin{pmatrix} 3! \\ 5 \end{pmatrix} - \begin{pmatrix} n-1 \\ r^*+1 \end{pmatrix}^3 + 11\begin{pmatrix} n-1 \\ r^*+1 \end{pmatrix}^2 - 14\begin{pmatrix} n-1 \\ r^*+1 \end{pmatrix}$$

which is the required proof. $\square$

Theorem 7: Let $N_i(i = \{0,1,2,\cdots\}) \in Z_n$ be finite non-negative integers, $\lambda_i = \{1,4,9,16,25,36,\ldots\}$ (set of perfect squares) where $n, \lambda_i \in \mathbb{N}$, then

$$F(n,r^*) = \lambda_i^{2*} + (n-2)$$

such that

$$|\Delta^*_{(n,r_3^*)}| = \lambda_i^{2*} + (n-2)$$

for all $n \geq 3 : N_3 = \{3,4,5,\cdots\}$.

Proof. Given any star-like transformation in $T\omega_n^*$, with $\Delta^*_{(n,r^*)}$ as star-like generator of all positive integers $(n-2)^2$ for all $n \geq 3 : N_3 = \{3,4,5,\cdots\}$, such that $r^* = 3$, where $1 = 1^2$, $4 = 2^2$, $9 = 3^2$; therefore the nth value of $\Delta^*_{(n,r^*)}$ can be given by $\sqrt{n} = n^2$ such that

$$n \in \lambda_i^2 : \lambda_1,\ \lambda_2 \ldots \lambda_n = \Delta^*_{(n,r^*)}$$

where $i \in \{1,4,9,25,\ldots \lambda_i n\}$. By the Theorem (6), we see that

$$|\Delta^*_{(n,r_3^*)}| = \lambda_i^2 + (n-2)$$

gives the recursive relation for the elements $\{2,18,84,260,\ldots\}$ of r^* for $N_3 = \{3,4,5,\cdots\} : r^* = 3$. The result follows immediately in Table 2. $\square$

Theorem 8: Given any star-like transformation $\alpha^* \in T\omega_n^*$ where $T\omega_n^* \subseteq P\omega_n^* \subseteq \alpha\omega_n^*$ is a star-like finite classical transformation, then $F(n,m^*)$ is reducible such that

$$\nabla^*_{(n,m^*)} = \frac{(n-1)^3 + 9(n-1)^2 - 22(n-1) + 18}{3!}$$

for all $n \geq 2 : N_2 = \{2,3,4,\cdots\}$ where $m^*(\alpha^*) = n-2$ is the maximum element in $T\omega_n^*$.

Proof. Let $\alpha^* \in T\omega_n^*$, then a finite cubic star-like classical polynomial $w_n = |F(n,m^*)|$ yields $\nabla^*_{(n,m^*)}$. Since $m^*(\alpha^*)$ has a unique positive integer difference one at 3rd order, which generates the set of w_n such that

$$|F(n,m^*)| = \Delta^*_{w_1}\begin{pmatrix} n \\ 1 \end{pmatrix} + \Delta^{2*}_{w_2}\begin{pmatrix} n \\ 2 \end{pmatrix} + \Delta^{3*}_{w_3}\begin{pmatrix} n \\ 3 \end{pmatrix}. \qquad (9)$$

If $n \geq 2 : N_2 = \{2,3,4,\cdots\}$ such that $m^*(\alpha^*) = n-2$, equation (6) and Theorem (6) give a star-like recurrence relation of $T\omega_n^*$ if

$$m^*(\alpha^*) = (n-2)$$

such that $\nabla^*_{(n,m^*)}$ where $m^* = n$. If $m^*(\alpha^*)$ is the maximum element and $T\omega_n^*$ is a star-like bijective, then $|m^*(\alpha^*)|$ can be specified in $\left(\begin{array}{c} n \\ m^* \end{array}\right)$ methods for every $n \geq m^* \geq 2 : N_2 = \{2,3,4,\cdots\}$. Thus,

$$\begin{aligned}
\nabla^*_{(n,m^*)} &= \frac{(m-1)^3+9(m-1)^2-22(m-1)+18}{3!} \\
\nabla^{2*}_{(n,\ m^*)} &= \frac{(m-1)^2+7(m-1)-4}{2!} \\
\nabla^{3*}_{(n,\ m^*)} &= m+3 \\
\nabla^{4*}_{(n,\ m^*)} &= \left(\begin{array}{c} n \\ m^* \end{array}\right)
\end{aligned}$$

such that

$$\nabla^*_{(n,m^*)} - \nabla^{2*}_{(n,m^*)} - \nabla^{3*}_{(n,m^*)} - \nabla^{4*}_{(n,m^*)} - \nabla^{5*}_{(n,m^*)} - \cdots\cdots - \nabla^{n+1*}_{(n,m^*)}.$$

□

Lemma 9: Let α_i^* be any star-like transformation. There are finitely many star-likes $\alpha_n^* \in T\omega_n^*$ such that:

$$F(n,r^*,m^*) = \left(\begin{array}{c} 2(n-2)+(n-m^*) \\ n-m^* \end{array}\right)$$

for $r^* \geq m^* \geq n \geq 1 : N_1 = \{1,2,3,\cdots\}$.

Proof. Suppose $\alpha_n^* \in T\omega_n^*$, then if α_n^* is bijective star-like mapping under composition of mapping where $F(n,r^*,m^*)$ is reducible, then, by Definition (2)

$$F(n,r^*,m^*) = \left(\begin{array}{c} 2(n-2)+(n-m^* \\ n-m^* \end{array}\right)$$

such that $\Delta^*_{n,r^*} = \nabla^*_{(n,m^*)}$ where $r^* \geq m^* \geq n \geq 1 : N_1 = \{1,2,3,\cdots\}$. Hence, the result follows immediately in Tables 2 and 3. □

Proposition 10: Let $\alpha^*_{(i,j)} \in P\omega_n^*$, let ν^* be the joint of any given star-like transformation, such that a non-negative integer $k \in w_i(n)$ exists, where $\nabla^{k+1^*}_{w_n} = t^*_{i(n)}$, then

$$2|\nu^*(\alpha^*_{(i,j)})| = \begin{pmatrix} (n-1)^3 \\ \nu^{3*}-1 \end{pmatrix} - \begin{pmatrix} 5(n-1)^2 \\ 5\nu^{2*}-1 \end{pmatrix} + \begin{pmatrix} 22(n-1) \\ 22\nu^*-1 \end{pmatrix} - \begin{pmatrix} 3!+2^3 \\ 2!+11 \end{pmatrix}$$

for all $n \geq 2$ and $\nu^* \geq 1 : N_{i(n\ j)}(i = j\{0,1,2,\cdots\})$

Proof. Suppose a non-negative integer $k \in Z_n$ exists such that $\nabla^{k+1*}_{w_n} = t^*_{i(n)}$. If $|\nu^*(\alpha^*_{(i,j)})|$ is the maximum convergence point of $w_1, w_2, \ldots w_n$ in $\nabla^*_{\nu^*(\alpha^*_{(i,j)})}$ such that the diagonal operator of $|\nu^*(\alpha^*_{(i,j)})|$ is $\nabla^*_{\nu^*(\alpha^*_{(i,j)})}$, whenever $n = \nu^* + 1$, then

$$w_n = \nabla^*_{w_1}\begin{pmatrix} n \\ 1 \end{pmatrix} + \nabla^{2*}_{w_2}\begin{pmatrix} n \\ 2 \end{pmatrix} + \nabla^{3*}_{w_3}\begin{pmatrix} n \\ \nu^* \end{pmatrix} \tag{10}$$

converges uniformly. Since the 3rd finite diagonal difference in equation (10) is constant, this gives a star-like cubic relation. By Lemma (1)

$$|\nu^*(\alpha^*_{(i,j)})| = F(n, \nu^*).$$

Then, under the composition of a star-like bijective transformation such that $\alpha^*_{(i,j)} \in D(\alpha^*_{(i,j)})$, $F(n,\nu^*)$ can be chosen in $\begin{pmatrix} n-1 \\ \nu^* \end{pmatrix}$ ways, which gives

$$2w_n = n^3 - 5n^2 + 22n - 14. \tag{11}$$

With simple calculation and in equation (11)

$$2\sum_{n=2}^{\nu^*-1} w_{n-1} = |\nu^*(\alpha^*_{(i,j)})|$$

then

$$2F(n,\nu^*) = \begin{pmatrix} (n-1)^3 \\ \nu^{3*}-1 \end{pmatrix} - \begin{pmatrix} 5(n-1)^2 \\ 5\nu^{2*}-1 \end{pmatrix} + \begin{pmatrix} 22(n-1) \\ 22\nu^*-1 \end{pmatrix} - \begin{pmatrix} 3!+2^3 \\ 2!+\ 11 \end{pmatrix}.$$

For any given $\alpha^*_{(i,j)} \in P\omega^*_n$ whenever $n = \nu^* + 1$, see that

$$\nabla^*_{\nu^*(\alpha^*_{(i,j)})} = F(n,\nu^*)$$

for all $n, \nu^* \in N_{i(n,\nu^*)}(i = \nu^*\{0,1,2,\cdots\})$ which is the required recursive relation of $|\nu^*(\alpha^*_{(i,j)})|$. □

Lemma 11: Let $d^*(\alpha^*_{(j,i)})$ be the star-like depth of any given $\alpha^*_{(j,i)} \in P\omega^*_n$, then $\Delta^{k+1*}_{w_n} = t^*_{i(n)}$ for a non-negative integer, and some star-like sequences $w_i(n)$, where $\alpha^*_{(j,i)} w_n \in D(\alpha^*_{(j,i)})$, gives

$$|d^*_n(\alpha^*_{(j,i)})| = F(n,\ d^*)$$

for all $n \geq 2 : N_{i(n\ j)}(i = \{0,1,2,\cdots\})$.

Proof. Suppose $\alpha^* \in \omega_n^*$, let $w_1, w_2, \ldots w_n$ be star-like sequence of numbers, and in Lemma (1), a non-negative integer $k \in Z_n$ exists such that $\Delta_{w_1}^{k+1*} = t_{i(n)}^*$. Then $|d^*(\alpha_{(j,i)}^*)|$ is the minimum convergence point of any given $\Delta^*(d_n^*)$ of $\alpha_{(j,i)}^* \in P\omega_n^*$ such that

$$F(n,\ d^*) = \begin{cases} \begin{pmatrix} 2^{n-1}+1 \\ 2^{n-1} \end{pmatrix}, \begin{pmatrix} n^2-n \\ d^{2*}-d^*+1 \end{pmatrix}; for\ n \leq d^* \leq 2 \\ \begin{pmatrix} n^2-n+1 \\ n(n-1) \end{pmatrix}, \begin{pmatrix} d^{2*}-1 \\ 1 \end{pmatrix}; n \geq d^* \geq 3 \end{cases}$$

□

Lemma 12: Let $O^{cd}P\omega_n^*$ be a star-like ordered connected partial semigroup. If $\alpha^* \in O^{cd}P\omega_n^*$ is a star-like transformation, there exist some non-negative integers such that $\Delta_{w_n}^{k+1*} = t_{i(n)}^*$ with star-like sequences $w_i(n) \in D(\alpha^*)$. Then

$$4!\ |O^{cd}| = \begin{cases} \begin{pmatrix} n^4 \\ t^{4*} \end{pmatrix} - 2!\begin{pmatrix} n^3 \\ t^{3*} \end{pmatrix} + \begin{pmatrix} n^2 \\ t^{2*} \end{pmatrix} + \begin{pmatrix} 2\times 4! \\ t_k^* \end{pmatrix}; n \leq t^* \leq 1 \\ \begin{pmatrix} n^4 \\ t^{4*}-1 \end{pmatrix} - 2!\begin{pmatrix} n^3 \\ t^{3*}-1 \end{pmatrix} + \begin{pmatrix} n^2 \\ t^{2*} \end{pmatrix}\begin{pmatrix} 59n-58 \\ t_k^* \end{pmatrix} + \begin{pmatrix} 2\times 4! \\ t_k^* \end{pmatrix}; \\ n \geq t^* \geq 1. \end{cases}$$

Proof. Given any star-like transformation $\alpha^* \in O^{cd}P\omega_n^*$, there exist finitely many star-like sequences $w_i(n)$ with some non-negative integers $k_i(n) \in Z_n)$ such that $\Delta_{w_n}^{k+1*} = t_{i(n)}^*$. Let $O^{cd} = O^{cd}P\omega_n^*$. If α^* is star-like bijective under the composition of star-like mapping, then

$$4!\ w_n = (\ n^4\ ) - (\ 2t^{3*}\ ) + (\ 59t^{2*}\ ) - (\ 58(t^*)\ ) + 48 \qquad (12)$$

for all $n \geq t^* \geq 1 : N_{i(n,t^*)}(i = \{0,1,2,\cdots\})$. In equation (12), $\Delta^* w_i(n)$ of $\alpha^* \in O^{cd}P\omega_n^*$ has a constant finite element in the 4th order, then for all $n \geq 1$ the star-like recursive relation of star-like $O^{cd}P\omega_n^*$ order connected transformation is obtained. □

TABLE 1. Size of $r^{+}(\alpha^{*})$ for some value of n

$\frac{n}{\lvert T\omega_n^*\rvert} = r^{+}(\alpha^{*})$	1	2	3	4	5	6	7	8	$\sum_{r^{+}(\alpha^{*})=0}^{n} \lvert r^{+}(\alpha^{*})\rvert$
1	1								1
2	1	2							3
3	1	7	2						10
4	1	16	18	2					37
5	1	33	84	31	2				151
6	1	66	260	99	43	2			471
7	1	131	630	269	84	51	2		1168
8	1	260	1302	623	125	100	52	2	2465

TABLE 2. Size of $F(n,m^{*})$ for some value of n

$\frac{n}{\lvert T\omega_n^*\rvert} = m^{*}(\alpha^{*})$	0	1	2	3	4	5	6	7	$\sum_{m^{*}=0}^{n} \lvert m^{*}(\alpha^{*})\rvert$
1	-	1							1
2	1	1	1						3
3	3	3	3	1					10
4	12	9	10	5	1				37
5	42	38	40	23	7	1			151
6	135	123	64	96	43	9	1		471
7	417	197	204	90	177	71	11	1	1168
8	1266	134	259	285	116	283	108	13	2465

TABLE 3. Size of $m^{*}(\alpha^{*})$ for some value of n

$\frac{n}{\lvert f(m^{*})\rvert} = m^{*}(\alpha^{*})$	0	1	2	3	4	5	$\sum_{m^{*}=0}^{k} F(n,m^{*})$
1	1	1					2
2	4	3	1				8
3	19	12	6	1			38
4	94	62	32	9	1		198
5	523	327	182	64	12	1	1109

TABLE 4. Size of $v^{*}(\alpha^{*})$ for some value of n

$\frac{n}{\lvert v^{*}(\alpha^{*})\rvert} = v^{*}(\alpha^{*})$	0	1	2	3	4	5	$\sum_{v^{*}=0}^{k} F(n,v^{*})$
1	2	0					2
2	5	2	1				8
3	15	14	9	0			38
4	49	62	69	17	1		198
5	167	307	401	205	29	0	1109

TABLE 5. Size of $t^*(\alpha^*)$ for some value of n

$\frac{n}{\lvert V^+(\alpha^*)\rvert} = t^*(\alpha^*)$	0	1	2	3	4	5	$\sum_{t^*=0}^{k} F(n,t^*)$
1	1	1					2
2	1	4	2				7
3	1	10	8	2			21
4	3	16	24	10	2		55
5	1	29	52	30	12	2	126

TABLE 6. Size of $v^*(\alpha^*)$ for some value of n

$\frac{n}{\lvert v^*(\alpha^*)\rvert} = v^*(\alpha^*)$	0	1	2	3	4	5	$\sum_{v^*=0}^{k} F(n,v^*)$
1	2	0					2
2	4	2	1				7
3	9	9	3	0			21
4	16	18	16	4	1		55
5	31	33	42	16	4	0	126

3. CONCLUSION

Finally, the work provides a useful investigation into the geometric structure of star-like transformation semigroups using combinatorial theory. Its novel approach and successful implementation of additional combinatorial functions make a significant contribution to the discipline. The output of the results is based on vertical and diagonal star-like order, creating an open problem wherein any given star-like classical finite transformation semigroup is star-like order-preserving-reversing.

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