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A NOVEL FIXED POINT ITERATION PROCESS APPLIED IN SOLVING DELAY DIFFERENTIAL EQUATIONS

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ABSTRACT. In this paper we introduce a new four-step iteration process, called the Picard-Noor hybrid iterative process, and prove that this new iteration scheme converges to the unique fixed point of contraction operators. We then show that it converges faster than Picard, Mann, Noor, Krasnoselskii, Ishikawa, Picard-Mann, Picard-Krasnoselskii, and Picard-Ishikawa iteration processes with numerical examples visualized in graphs and tables to substantiate our claims. Data dependence result is proved and stability of the new process is shown. Our result is applied to the approximation of the solution of delay differential equations. Our results generalize and extend other results in literature.

1. INTRODUCTION

Fixed point theory has been very remarkable since its introduction and it has evolved greatly by hosting a great number of researches of many sort. Several nonlinear problems have been solved by the fixed point theory method by transforming any of such problem into an operator equation of the form;

$$Tx = x, \tag{1}$$

where T is a self-map in Banach spaces. The solution of the resulting equation is approximated using a suitable fixed point iteration process. By this approach, the solution of many nonlinear problems have been approximated (see, e.g. [21, 22]). For example, many authors have used the fixed point iteration process to approximate the solution of delay differential equations (see [12], [20],[9],[18]) amongst other class of nonlinear problems.

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It is our aim in this paper to construct a hybrid fixed point iteration process of Picard [23] and Noor [17] that generalizes and extends other existing results for a contraction mapping in a uniformly convex Banach space and use it to approximate the solution of delay differential equations.

The rest of this paper is arranged such that Section 2 is for preliminaries, where preliminary definitions and lemmas are presented. Section 3 considers the main results that comprises of convergence of the new iterative process to a unique fixed point of a contractive operator. The rate of convergence of the Picard-Noor process viz a viz other existing iteration processes is explored. Data dependence result and stability of the new process is considered in Section 4. In Section 5, our new iterative process is applied to the approximation of the solution of delay differential equations. Furthermore, Section 6 presents the concluding remarks of this paper.

2. PRELIMINARIES

Let C be a nonempty convex subset of a normed space $\mathscr{X}$, and let $T : C \to C$ be a self mapping. The point $x \in \mathscr{X}$ is said to be a fixed point of T if (1) holds and $\mathfrak{F}(T) \neq \emptyset$, where $\mathfrak{F}(T)$ denotes the set of all fixed points of T. For any $x_0 \in \mathscr{X}$, the sequence $\{x_n\}_{n\geq 0} \subset \mathscr{X}$ given by

$$x_n = Tx_{n-1} = T^n x_0, n = 1, 2, \ldots \tag{2}$$

is called the sequence of successive approximations or the Picard iteration starting at x_0. In the sequel, $\mathbb{N}$ will denote the set of all natural numbers.

The Picard iteration process was introduced by Picard [23] and it is defined by the sequence $\{u_n\}_{n=0}^{\infty}$ as follows:

$$\begin{cases} u_1 = u \in C, \\ u_{n+1} = Tu_n, n \in \mathbb{N} \end{cases} \tag{3}$$

In 1953, Mann [15] introduced the Mann iterative process defined as:

$$\begin{cases} v_1 = v \in C, \\ v_{n+1} = (1-\alpha_n)v_n + \alpha_n T v_n, n \in \mathbb{N}, \end{cases} \tag{4}$$

where $\{\alpha_n\}_{n=0}^{\infty} \in (0,1)$.

Genel [10] in 1975 showed that the Mann iterative process has only weak convergence even in a Hilbert space. Modifications have been recently made on Mann iteration scheme to guarantee a strong convergence. In 2002, Moore [16] introduced the idea of a double-sequence iteration, and proved that a Mann-type double-sequence iteration process converges strongly to a fixed point of a continuous pseudocontractive map. The Krasnoselskii [13] iteration process is defined by:

$$\begin{cases} s_1 \in C, \\ s_{n+1} = (1-\lambda)s_n + \lambda T s_n, n \geq 0, \end{cases} . \tag{5}$$

where $\lambda \in (0,1)$ is the iterative parameter.
In 1974, Ishikawa [11] also introduced the iteration process given by the sequence $\{z_n\}_{n=0}^{\infty}$:

$$\begin{cases} z_1 = z \in C, \\ z_{n+1} = (1-\alpha_n)z_n + \alpha_n T y_n, \\ y_n = (1-\beta_n)z_n + \beta_n T z_n, n \in \mathbb{N}, \end{cases} \tag{6}$$

where $\{\alpha_n\}_n^{\infty}$ and $\{\beta_n\}_n^{\infty} \in (0,1)$.
In 2000, Noor [17] introduced a three-step approximation schemes for general variational inequalities. The iteration process is defined by the sequence $\{t_n\}$:

$$\begin{cases} t_1 = t \in C, \\ t_{n+1} = (1-\alpha_n)t_n + \alpha_n T y_n, \\ y_n = (1-\beta_n)t_n + \beta_n T z_n, \\ z_n = (1-\gamma_n)t_n + \gamma_n T t_n, n \in \mathbb{N}, \end{cases} \tag{7}$$

where $\{\alpha_n\}_n^{\infty}, \{\beta_n\}_n^{\infty}, and \{\gamma_n\}_n^{\infty} \in (0,1)$.
Khan [14] in 2013 introduced an iteration process, called the Picard-Mann hybrid iterative process, which is a hybrid of Picard and Mann iterative processes and defined thus;

$$\begin{cases} m_1 = m \in C, \\ m_{n+1} = T z_n, \\ z_n = (1-\alpha_n)m_n + \alpha_n T m_n, n \in \mathbb{N}, \end{cases} \tag{8}$$

where $\{\alpha_n\}_n^{\infty} \in (0,1)$.
The author showed that the process converges faster than all of Picard, Mann and Ishikawa iterative processes in the sense of Berinde for contractions.
Thakur *et al* [27] in 2016 introduced a new iteration process, given by

the sequence $\{x_n\}$ defined as:

$$\begin{cases} x_{n+1} = (1-\alpha_n)Tx_n + \alpha_n Ty_n, \\ y_n = (1-\beta_n)z_n + \beta_n Tz_n, \\ z_n = (1-\gamma_n)x_n + \gamma_n Tx_n, \end{cases} \tag{9}$$

where $x_1 \in C$, and $\{\alpha_n\}, \{\beta_n\}$, and $\{\gamma_n\}$ are real sequences in (0,1). They showed that their process converges faster than all of Picard, Mann, Ishikawa, Noor, Agarwal et al., and Abbas et al. iteration processes for contractions in the sense of Berinde.

In 2014, Abbas and Nazir [1] in bit to answer the question, "*Is it possible to develop an iteration process whose rate of convergence is even faster than the Agarwal et al [2] iteration process?"*, introduced iteration process defined by the sequence $\{y_n\}$ as follows:

$$\begin{cases} y_1 = y \in C, \\ y_{n+1} = (1-\alpha_n)Tw_n + \gamma_n Tz_n, \\ w_n = (1-\beta_n)Ty_n + \alpha_n Tz_n, \\ z_n = (1-\gamma_n)y_n + \beta_n Ty_n, n \in \mathbb{N}, \end{cases} \tag{10}$$

where $\{\alpha_n\}_n^\infty, \{\beta_n\}_n^\infty$, and $\{\gamma_n\}_n^\infty \in (0,1)$.

Okeke [18] in 2017 introduced the Picard–Krasnoselskii hybrid iteration process which is a hybrid of Picard and Krasnoselskii iteration processes and showed that for nonlinear contractive operators, the iteration process converges faster than all of Picard, Mann, Krasnoselskii and Ishikawa iteration processes in the sense of Berinde. The scheme is defined by:

$$\begin{cases} y_1 = y \in C, \\ y_{n+1} = Tw_n, \\ w_n = (1-\lambda)y_n + \lambda Ty_n, n \in \mathbb{N}, \end{cases} \tag{11}$$

where $\lambda \in (0,1)$.

Moreover, Okeke [19] in 2019 introduced a hybrid of Picard and Ishikawa iteration processes called the Picard–Ishikawa hybrid iterative process, and showed that it converges faster than all of Picard, Krasnoselskii, Mann, Ishikawa, Noor, Picard–Mann and Picard–Krasnoselskii iteration processes in the sense of Berinde [3]. The iterative process is defined by:

$$\begin{cases} g_1 = y \in C, \\ g_{n+1} = Tw_n, \\ w_n = (1-\alpha_n)g_n + \alpha_n Tz_n, \\ z_n = (1-\beta_n)g_n + \beta_n Tg_n, n \in \mathbb{N}, \end{cases} \tag{12}$$

where $\{\alpha_n\}_n^\infty, and \{\beta_n\}_n^\infty$, are sequences $\in (0,1)$.

Recently in 2022, Jia *et al* [12] introduced a hybrid Picard-Thakur iteration process defined thus,

$$\begin{cases} j_1 = j \in C \\ j_{n+1} = Tk_n \\ k_n = (1-\alpha_n)Tm_n + \alpha_n Tl_n \\ l_n = (1-\beta_n)m_n + \beta_n Tm_n \\ m_n = (1-\gamma_n)j_n + \gamma_n Tj_n, n \in \mathbb{N} \end{cases} \tag{13}$$

for $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \in (0,1)$.

Motivated by the results above, we introduce a new iteration process called the Picard-Noor hybrid iteration process define as follows;

$$\begin{cases} x_1 = x \in C, \\ x_{n+1} = Tv_n, \\ v_n = (1-\alpha_n)x_n + \alpha_n Tu_n, \\ u_n = (1-\beta_n)x_n + \beta_n Tt_n, \\ t_n = (1-\gamma_n)x_n + \gamma_n Tx_n, \qquad n \in \mathbb{N} \end{cases} \tag{14}$$

where $\{\alpha\}$, $\{\beta\}$, $\{\gamma\}$ are real sequences in $(0,1)$.
We now state some definitions and lemmas that are crucial to the proof of the main results.

Definition 2.1. [3] Let C be a nonempty convex subset of a normed space $\mathscr{X}$, and let $T : C \to C$ be a mapping. T is called a contraction map if

$$\|Tx - Ty\| \leq \delta\|x-y\|, \quad \delta \in [0,1), \tag{15}$$

for all $x, y \in C$.

Definition 2.2. [6] Let $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ be two sequences of real numbers converging to a and b respectively. If

$$\lim_{n\to\infty} \frac{\|a_n - a\|}{\|b_n - b\|} = 0, \tag{16}$$

then $\{a_n\}$ is said to converge to a faster than $\{b_n\}$ to b.

Definition 2.3. [6] Suppose that for two fixed-point iterative processes $\{u_n\}_{n=0}^{\infty}$ and $\{v_n\}_{n=0}^{\infty}$, both converging to the same fixed point p, the error estimates

$$\|u_n - p\| \leq a_n, \text{ for all } n \in \mathbb{N},$$
$$\|v_n - p\| \leq b_n, \text{ for all } n \in \mathbb{N},$$

exist, where $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ are two sequences of positive numbers converging to zero. If $\{a_n\}_{n=0}^{\infty}$ converges faster than $\{b_n\}_{n=0}^{\infty}$ then $\{u_n\}_{n=0}^{\infty}$ converges faster than $\{v_n\}_{n=0}^{\infty}$ to p.

Definition 2.4. [7] Let T , $\widetilde{T}: C \to C$ be two operators. We say that $\widetilde{T}$ is an approximate operator of T if for all $x \in C$ and for a fixed $\varepsilon > 0$ we have
$\|Tx - \widetilde{T}x\| \leq \varepsilon$.

Whenever we talk of rate of convergence in this paper, we mean the one due to Berinde [4].

Lemma 2.5. [4] If ς is a real number such that $0 \leq \varsigma < 1$ and $\{\varepsilon_n\}_{n=0}^{\infty}$ is a sequence of positive numbers such that $\lim_{n\to\infty} \varepsilon_n = 0$, then for any sequence of positive numbers $\{r_n\}_{n=0}^{\infty}$ satisfying

$$r_{n+1} \leq \varsigma r_n + \varepsilon_n, \qquad n = 0, 1, 2, \ldots, \tag{17}$$

one has $\lim_{n\to\infty} r_n = 0$.

Lemma 2.6. [26] Let $\{s\}_{n=0}^{\infty}$ be a sequence of positive real numbers including zero satisfying,

$$s_{n+1} \leq (1 - \mu_n) s_n. \tag{18}$$

If $\{\mu_n\}_{n=0}^{\infty} \subset (0,1)$ and $\sum_{n=0}^{\infty} \mu_n = \infty$, then $\lim_{n\to\infty} s_n = 0$.

Lemma 2.7. [28] Let $\{\beta\}_{n=0}^{\infty}$ be a nonnegative sequence satisfying

$$\beta_{n+1} \leq (1 - \lambda_n)\beta_n + \rho_n, \tag{19}$$

with $\lambda_n \in [0,1]$, $\sum_{i=1}^{\infty} \lambda_i = \infty$, and $\rho_n = o(\lambda_n)$. Then $\lim_{n\to\infty} \beta_n = 0$.

Lemma 2.8. [25] Let $\{\lambda_n\}_n^{\infty}$ be a nonnegative sequence such that there exist $n_0 \in \mathbb{N}$, and such that for all $n \geq n_0$ the inequality is satisfied,

$$\rho_{n+1} \leq (1 - \zeta_n)\rho_n + \zeta_n \eta_n, \tag{20}$$

where $\zeta_n \in (0,1)$, $\sum_{n=0}^{\infty} \zeta_n = \infty$, and $\eta_n \geq 0$, for all $n \in \mathbb{N}$. Then the following inequality holds,

$$0 \leq \limsup_{n\to\infty} \rho_n \leq \limsup_{n\to\infty} \eta_n \tag{21}$$

In following sections, we prove some convergence theorems of the Picard-Noor hybrid iterative process.

3. Convergence analysis of the Picard-Noor hybrid iteration process

We now show that the Picard-Noor hybrid iterative process converges to a unique fixed point of contractive-like operators.

Theorem 3.1. Let C be a nonempty closed convex subset of a Banach space $\mathscr{X}$ and $T : C \to C$ be a contraction map satisfying the contractive condition (15) such that $\mathfrak{F}(T) \neq \emptyset$. Let $\{x\}_{n=1}^{\infty}$ be an iteration process generated by the Picard-Noor hybrid iteration process (14) with real sequences $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \in (0,1)$ satisfying $\sum_{k=0}^{\infty} \alpha_k = \infty$. Then $\{x\}_{n=1}^{\infty}$ converges to a unique fixed point of T, say $x^* \in \mathfrak{F}(T)$.

Proof. Banach's fixed point theorem guarantees the existence and the uniqueness of fixed point, x^*. We only need to show that Picard-Noor hybrid iterative process converges to x^*, i.e. $x_n \to x^*$ as $n \to \infty$.
Using (14) and (15), we have,

$$\begin{aligned}
\|t_n - x^*\| =& \|(1-\gamma_n)x_n + \gamma_n T x_n - x^*\| \\
\leq& (1-\gamma_n)\|x_n - x^*\| + \gamma_n \|T x_n - x^*\| \\
=& (1-\gamma_n)\|x_n - x^*\| + \gamma_n \|T x_n - T x^*\| \\
\leq& (1-\gamma_n)\|x_n - x^*\| + \delta \gamma_n \|x_n - x^*\| \\
=& [1-\gamma_n(1-\delta)]\|x_n - x^*\| \qquad (22)
\end{aligned}$$

$$\begin{aligned}
\|u_n - x^*\| =& \|(1-\beta_n)x_n + \beta_n T t_n - x^*\| \\
=& \|(1-\beta_n)(x_n - x^*) + \beta_n (T t_n - x^*)\| \\
\leq& (1-\beta_n)\|x_n - x^*\| + \beta_n \|T t_n - x^*\| \\
=& (1-\beta_n)\|x_n - x^*\| + \beta_n \|T t_n - T x^*\| \\
\leq& (1-\beta_n)\|x_n - x^*\| + \delta \beta_n \|t_n - x^*\| \\
\leq& (1-\beta_n)\|x_n - x^*\| + \delta \beta_n [1-\gamma_n(1-\delta)]\|x_n - x^*\| \\
=& [1-\beta_n(1-\delta[1-\gamma_n(1-\delta)])]\|x_n - x^*\| \\
\leq& [1-\beta_n(1-\delta)]\|x_n - x^*\| \qquad (23)
\end{aligned}$$

Using (14) and (23)

$$\begin{aligned}
\|v_n - x^*\| =& \|(1-\alpha_n)x_n + \alpha_n T u_n - x^*\| \\
=& \|(1-\alpha_n)(x_n - x^*) + \alpha_n (T u_n - x^*)\| \\
\leq& (1-\alpha_n)\|x_n - x^*\| + \alpha_n \|T u_n - x^*\| \\
\leq& (1-\alpha_n)\|x_n - x^*\| + \delta \alpha_n \|u_n - x^*\| \\
\leq& [1-\alpha_n(1-\delta[1-\beta_n(1-\delta)])]\|x_n - x^*\| \qquad (24)
\end{aligned}$$

Finally, using (14) and (24), we have

$$\begin{aligned}
\|x_{n+1} - x^*\| =& \|T v_n - x^*\| \\
\leq& \|T v_n - T x^*\| \\
\leq& \delta \|v_n - x^*\| \\
\leq& \delta [1-\alpha_n(1-\delta[1-\beta_n(1-\delta)])]\|x_n - x^*\|
\end{aligned}$$

Since $\beta_n, \gamma_n \in [0,1]$ and $\delta \in [0,1)$, then

$$\|x_{n+1} - x^*\| \leq \delta[1 - \alpha_n(1-\delta)]\|x_n - x^*\|.$$

Inductively,

$$\|x_1 - x^*\| \leq \delta[1 - \alpha_n(1-\delta)]\|x_0 - x^*\|.$$

So that

$$\|x_{n+1} - x^*\| \leq \delta^{n+1}\|x_0 - x^*\| \prod_{k=0}^{n}[1 - \alpha_k(1-\delta)].$$

Recalling that $\alpha_n \in [0,1]$ and $0 \leq \delta < 1$ such that $1 - \alpha_n(1-\delta) < 1$ for all $n \in \mathbb{N}$.
Clearly, from elementary analysis, $1 - x \leq e^{-x}$ for $x \in [0,1]$.
Therefore,

$$\begin{aligned}\|x_{n+1} - x^*\| &\leq \delta^{(n+1)}\|x_0 - x^*\| \prod_{k=0}^{n} e^{-(1-\delta)\alpha_k} \\ &\vdots \\ &\leq \delta^{(n+1)}\|x_0 - x^*\|^{n+1} e^{-(1-\delta)\sum_{k=0}^{n}\alpha_k}.\end{aligned}$$

Moreover, we have from the hypothesis that $\sum_{k=0}^{\infty}\alpha_k = \infty$,
so that $e^{-(1-\delta)\sum_{k=0}^{n}\alpha_k} \to 0$ as $n \to \infty$ and consequently, $\lim_{n\to\infty}\|x_n - x^*\| = 0$.
Hence, the proof is complete. □

Now we shall prove that the Picard-Noor hybrid iterative process converges faster than the Picard-Ishiwaka hybrid iteration in the sense of Berinde 2.5.

Theorem 3.2. Let C be a nonempty closed convex subset of a Banach space $\mathscr{X}$ and $T : C \to C$ be a contraction mapping satisfying the contraction condition (15) such that $\mathfrak{F}(T) \neq \emptyset$. Let $\{g_n\}$ and $\{x_n\}$ be two iterative sequences defined by the Picard-Ishiwaka hybrid and the Picard-Noor hybrid iteration processes respectively, converging to a fixed point $x^* \in \mathfrak{F}(T)$. Suppose $y_1 = x_1 \in C$, then the Picard-Noor hybrid iterative process converges faster than the Picard-Krasnoselskii hybrid iteration process.

Proof. Suppose $x^* \in F(T)$. Using (11) and (15),

$$\begin{aligned}\|w_n - x^*\| =& \|(1-\lambda)y_n + \lambda T y_n - x^*\| \\ \leq& (1-\lambda)\|y_n - x^*\| + \lambda\|Ty_n - x^*\| \\ \leq& (1-\lambda)\|y_n - x^*\| + \lambda\delta\|y_n - x^*\| \\ =& [1 - \lambda(1-\delta)]\|y_n - x^*\|\end{aligned}$$

$$\begin{aligned}\|y_{n+1}-x^*\| =&\|Tw_n-x^*\|\\ \leq&\delta\|w_n-x^*\|\\ \leq&\delta[1-\lambda(1-\delta)]\|y_n-x^*\|.\end{aligned}$$

Inductively,

$$\|y_1-x^*\| \leq \delta[1-\lambda(1-\delta)]\|y_0-x^*\| \tag{25}$$

From (25), we have,

$$\|y_{n+1}-x^*\| \leq \delta^{n+1}[1-\lambda(1-\delta)]^{n+1}\|y_0-x^*\|.$$

Moreover, from proof of Theorem 3.1, we have

$$\begin{aligned}\|x_{n+1}-x^*\| \leq&\delta^{n+1}\|x_0-x^*\|\prod_{k=0}^{n}[1-\alpha_k(1-\delta)]\\ =&\delta^{(n+1)}\|x_0-x^*\|[1-\alpha_k(1-\delta)]^{n+1}\\ \leq&\delta^{(n+1)}\|x_0-x^*\|[1-\alpha(1-\delta)]^{n+1}\end{aligned}$$

Let

$$a_n = \delta^{(n+1)}[1-\alpha(1-\delta)]^{n+1}\|x_0-x^*\|$$

$$b_n = \delta^{n+1}[1-\lambda(1-\delta)]^{n+1}\|y_0-x^*\|$$

and

$$\Omega_n = \frac{a_n}{b_n} = \frac{[1-\alpha(1-\delta)]^{n+1}\|x_0-x^*\|}{[1-\lambda(1-\delta)]^{n+1}\|y_0-x^*\|} \to 0 \ \ as \ \ n\to\infty,$$

Thus, Picard-Noor hybrid iteration process converges faster than Picard-Krasnoselskii hybrid iteration process. Therefore completing the proof. □

To support our analytical results, we now give a numerical example.

Example 3.3. Let $C=[1,10]\subseteq X=\mathbb{R}$ and $T:C\to C$ be an operator defined by $Tx=\sqrt[3]{4x+15}$ for all $x\in C$. Choose $\alpha_n=\beta_n=\gamma_n=\frac{2}{3}$ and $\lambda=\frac{1}{2}$ for each $n\in\mathbb{N}$ with the initial value $x_0=5$. T is a contraction mapping with contractive constant $\delta=\frac{1}{\sqrt[3]{15}}$ and the set of fixed points, $\mathfrak{F}(T)=\{3\}$.

The table and the graph below are generated with MATLAB R2015a. From the Table 1 and Table 2, it is obvious that our iteration process (14) converges faster than all of Picard–Krasnoselskii, Picard, Mann, Krasnoselskii, Ishikawa, and Noor iteration processes.

TABLE 1. Comparison of the rate of convergence of several iteration process for Example 1

Step	Picard-Noor	Picard-Krasnoselskii	Noor	Ishikawa
1	5.0000000000	5.0000000000	5.0000000000	5.0000000000
2	3.1045990422	3.1595872391	3.7309467974	3.7402882565
3	3.0056747910	3.0134981297	3.2678221948	3.2751109448
4	3.0003084921	3.0011474515	3.0982286274	3.1024052248
5	3.0000167720	3.0000975846	3.0360405479	3.0381425261
6	3.0000009119	3.0000082993	3.0132252714	3.0142101919
7	3.0000000496	3.0000007058	3.0048533304	3.0052945502
8	3.0000000027	3.0000000600	3.0017810791	3.0019727526
9	3.0000000001	3.0000000051	3.0006536264	3.0007350579
10	3.0000000000	3.0000000004	3.0002398706	3.0002738877
11	3.0000000000	3.0000000000	3.0000880288	3.0001020526
12	3.0000000000	3.0000000000	3.0000323052	3.0000380256
13	3.0000000000	3.0000000000	3.0000118555	3.0000141686
14	3.0000000000	3.0000000000	3.0000043508	3.0000052793
15	3.0000000000	3.0000000000	3.0000015967	3.0000019671
16	3.0000000000	3.0000000000	3.0000005860	3.0000007330
17	3.0000000000	3.0000000000	3.0000002150	3.0000002731
18	3.0000000000	3.0000000000	3.0000000789	3.0000001018
19	3.0000000000	3.0000000000	3.0000000290	3.0000000379
20	3.0000000000	3.0000000000	3.0000000106	3.0000000141

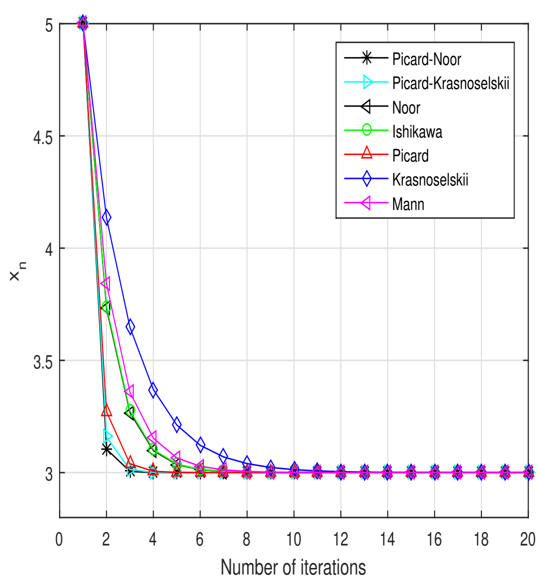

FIGURE 1. Graph corresponding to Table 1 and Table 2

TABLE 2. Comparison of the rate of convergence of several iteration process for Example 1

Step	Picard	Krasnoselskii	Mann
1	5.0000000000	5.0000000000	5.0000000000
2	3.2710663102	4.1355331551	3.8416144278
3	3.0396320986	3.6475601971	3.3580210177
4	3.0058599681	3.3702904719	3.1530734262
5	3.0008678923	3.2120873444	3.0655947572
6	3.0001285711	3.1215921195	3.0281361796
7	3.0000190475	3.0697493363	3.0120738442
8	3.0000028218	3.0400235912	3.0051820956
9	3.0000004181	3.0229706656	3.0022243309
10	3.0000000619	3.0131849371	3.0009547904
11	3.0000000092	3.0075684953	3.0004098482
12	3.0000000014	3.0043446675	3.0001759303
13	3.0000000002	3.0024940920	3.0000755196
14	3.0000000000	3.0014317708	3.0000324174
15	3.0000000000	3.0008219350	3.0000139155
16	3.0000000000	3.0004718491	3.0000059733
17	3.0000000000	3.0002708755	3.0000025641
18	3.0000000000	3.0001555023	3.0000011007
19	3.0000000000	3.0000892698	3.0000004725
20	3.0000000000	3.0000512474	3.0000002028

Example 3.4. Let $C = [1,6] \subseteq X = \mathbb{R}$ and $T : C \to C$ be an operator defined by $Tx = \frac{x}{2} + 1$ for all $x \in C$. Choose $\alpha_n = \frac{1}{2}$, $\beta_n = \frac{1}{3}$, $\gamma_n = \frac{1}{4}$, $\lambda = \frac{1}{2}$ for each $n \in \mathbb{N}$ with the initial value $x_1 = 2.5$. T is a contraction mapping and the set of fixed points, $\mathfrak{F}(T) = \{2\}$.

From the Table 3 and Table 4, it is clear that our iteration process (14) converges faster than all of Picard-Ishikawa, Picard-Mann, Picard-Krasnoselskii, Picard, Mann, Krasnoselskii, Ishikawa, and Noor iterative processes. Figure 2 is the plot of Table 3 and Table 4.

We now show equivalence in convergence of the Picard-Noor and the Abbas-Nazir iterative processes.

Theorem 3.5. Let C be a nonempty closed convex subset of a Banach space $\mathscr{X}$ and $T : C \to C$ a contraction mapping satisfying the contraction condition (15) such that $\mathfrak{F}(T) \neq \emptyset$. Let $\{j_n\}$ and $\{x_n\}$ be two iterative sequences defined by the Picard-Thakur iteration process (13) and the Picard-Noor hybrid iteration process (14) respectively, with real sequences $\alpha_n, \beta_n, \gamma_n \in (0,1)$. Then the following are equivalent:

TABLE 3. Comparison of the rate of convergence of several iteration process for Example 2

Steps	Picard-Noor	Picard-Mann	Noor	Mann
1	2.5000000000	2.5000000000	2.5000000000	2.5000000000
2	2.1639453125	2.2087500000	2.3278906250	2.3750000000
3	2.0537561310	2.0871531250	2.2150245239	2.2812500000
4	2.0176261314	2.0363864297	2.1410090511	2.2109375000
5	2.0057794432	2.0151913344	2.0924710918	2.1582031250
6	2.0018950253	2.0063423821	2.0606408082	2.1186523438
7	2.0006213610	2.0026479445	2.0397671050	2.0889892578
8	2.0002037385	2.0011055168	2.0260785218	2.0667419434
9	2.0000668039	2.0004615533	2.0171018056	2.0500564575
10	2.0000219044	2.0001926985	2.0112150435	2.0375423431
11	2.0000071822	2.0000804516	2.0073546152	2.0281567574
12	2.0000023550	2.0000335886	2.0048230188	2.0211175680
13	2.0000007722	2.0000140232	2.0031628453	2.0158381760
14	2.0000002532	2.0000058547	2.0020741346	2.0118786320
15	2.0000000830	2.0000024443	2.0013601786	2.0089089740
16	2.0000000272	2.0000010205	2.0008919796	2.0066817305
17	2.0000000089	2.0000004261	2.0005849435	2.0050112979
18	2.0000000029	2.0000001779	2.0003835950	2.0037584734
19	2.0000000010	2.0000000743	2.0002515544	2.0028188551
20	2.0000000003	2.0000000310	2.0001649647	2.0021141413
21	2.0000000001	2.0000000129	2.0001081807	2.0015856060
22	**2.0000000000**	2.0000000054	2.0000709429	2.0011892045
23	2.0000000000	2.0000000023	2.0000465230	2.0008919034
24	2.0000000000	2.0000000009	2.0000305089	2.0006689275
25	2.0000000000	2.0000000004	2.0000200072	2.0005016956

(1) $\lim_{n\to\infty}\|x_n - x^*\| = 0$;
(2) $\lim_{n\to\infty}\|j_n - x^*\| = 0$

Proof. Let $x^* \in \mathfrak{F}(T) \neq \emptyset$. We now prove that (1) $\Longrightarrow$ (2), that is, if Picard-Noor hybrid iterative process (14) converges to x^*, i.e. $\lim_{n\to\infty}\|x_n - x^*\| = 0$, then the Picard-Thakur iteration process (13) also converges, i.e. $\lim_{n\to\infty}\|j_n - x^*\| = 0$.
Using (14) and (10), and Lemma 2.7:

$$\begin{aligned}\|t_n - m_n\| =&\|(1-\gamma_n)x_n + \gamma_n T x_n - (1-\gamma_n)j_n - \gamma_n T j_n\|\\ \leq&(1-\gamma_n)\|x_n - j_n\| + \gamma_n\|T x_n - T j_n\|\\ \leq&(1-\gamma_n)\|x_n - j_n\| + \gamma_n\delta\|x_n - j_n\|\\ =&[1-\gamma_n(1-\delta)\|x_n - j_n\|] \end{aligned} \tag{26}$$

TABLE 4. Comparison of the rate of convergence of several iteration process for Example 2

Steps	Picard-Krasnoselskii	Picard-Ishikawa	Ishikawa	Krasnoselskii
1	2.5000000000	2.5000000000	2.5000000000	2.5000000000
2	2.1875000000	2.1771875000	2.3543750000	2.3750000000
3	2.0703125000	2.0627908203	2.2511632813	2.2812500000
4	2.0263671875	2.0222514969	2.1780119756	2.2109375000
5	2.0098876953	2.0078853742	2.1261659877	2.1582031250
6	2.0037078857	2.0027943795	2.0894201438	2.1186523438
7	2.0013904572	2.0009902582	2.0633765269	2.0889892578
8	2.0005214214	2.0003509228	2.0449181134	2.0667419434
9	2.0001955330	2.0001243583	2.0318357129	2.0500564575
10	2.0000733249	2.0000440695	2.0225635615	2.0375423431
11	2.0000274968	2.0000156171	2.0159919242	2.0281567574
12	2.0000103113	2.0000055343	2.0113342763	2.0211175680
13	2.0000038667	2.0000019612	2.0080331683	2.0158381760
14	2.0000014500	2.0000006950	2.0056935081	2.0118786320
15	2.0000005438	2.0000002463	2.0040352738	2.0089089740
16	2.0000002039	2.0000000873	2.0028600003	2.0066817305
17	2.0000000765	2.0000000309	2.0020270252	2.0050112979
18	2.0000000287	2.0000000110	2.0014366541	2.0037584734
19	2.0000000108	2.0000000039	2.0010182286	2.0028188551
20	2.0000000040	2.0000000014	2.0007216695	2.0021141413
21	2.0000000015	2.0000000005	2.0005114833	2.0015856060
22	2.0000000006	2.0000000002	2.0003625138	2.0011892045
23	2.0000000002	2.0000000001	2.0002569316	2.0008919034
24	2.0000000001	**2.0000000000**	2.0001821003	2.0006689275
25	**2.0000000000**	2.0000000000	2.0001290636	2.0005016956

$$\begin{aligned}\|u_n - l_n\| =& \|(1-\beta_n)x_n + \beta_n T t_n - (1-\beta_n)m_n - \beta_n T m_n\| \\ \leq& (1-\beta_n)\|x_n - m_n\| + \beta_n \|T t_n - T m_n\| \\ \leq& (1-\beta_n)\|x_n - m_n\| + \beta_n \delta \|t_n - m_n\| \\ \leq& (1-\beta_n)\|x_n - m_n\| + \beta_n \delta \{[1-\gamma_n(1-\delta)]\|x_n - j_n\|\} \end{aligned} \tag{27}$$

$$\begin{aligned}\|x_n - m_n\| =& \|x_n - (1-\gamma_n) j_n - \gamma_n T j_n\| \\ \leq& (1-\gamma_n)\|x_n - j_n\| + \gamma_n \|x_n - T j_n\| \\ =& (1-\gamma_n)\|x_n - j_n\| + \gamma_n \|x_n - T x_n + T x_n - T j_n\| \\ \leq& (1-\gamma_n)\|x_n - j_n\| + \gamma_n \|x_n - T x_n\| + \gamma_n \delta \|x_n - j_n\| \\ =& [1-\gamma_n(1-\delta)]\|x_n - j_n\| + \gamma_n \|x_n - T x_n\|. \end{aligned} \tag{28}$$

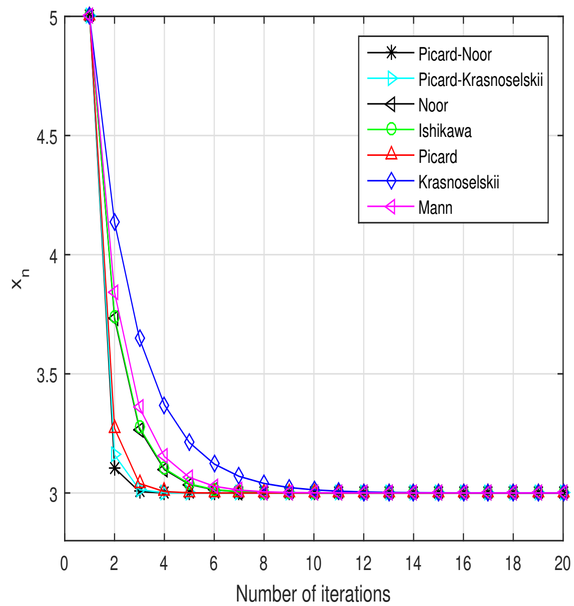

FIGURE 2. Graph corresponding to Table 3 and Table 4

Putting (28) in (27)

$$\begin{aligned}\|u_n - l_n\| \leq &(1-\beta_n)[1-\gamma_n(1-\delta)]\|x_n - j_n\| + (1-\beta_n)\gamma_n\|x_n - Tx_n\| \\ &+ \beta_n\delta[1-\gamma_n(1-\delta)]\|x_n - j_n\| \\ = &\{(1-\beta_n)[1-\gamma_n(1-\delta)] + \beta_n\delta[1-\gamma_n(1-\delta)]\}\|x_n - j_n\| \\ &+ (1-\beta_n)\gamma_n\|x_n - Tx_n\| \\ = &[1-\beta_n(1-\delta)][1-\gamma_n(1-\delta)]\|x_n - j_n\| + (1-\beta_n)\gamma_n\|x_n - Tx_n\| \end{aligned} \tag{29}$$

$$\begin{aligned}\|v_n - k_n\| = &\|(1-\alpha_n)x_n + \alpha_n Tu_n - (1-\alpha_n)Tm_n - \alpha_n Tl_n\| \\ \leq &(1-\alpha_n)\|x_n - Tm_n\| + \alpha_n\|Tu_n - Tl_n\| \\ \leq &(1-\alpha_n)\|x_n - Tx_n + Tx_n - Tm_n\| + \alpha_n\delta\|u_n - l_n\| \\ \leq &(1-\alpha_n)\|x_n - Tx_n\| + (1-\alpha_n)\delta\|x_n - m_n\| + \alpha_n\delta\|u_n - l_n\| \end{aligned} \tag{30}$$

Combining (28), (29) and (30)

$$\begin{aligned}\|v_n-k_n\| \leq&(1-\alpha_n)\|x_n-Tx_n\|+(1-\alpha_n)\delta\Big\{[1-\gamma_n(1-\delta)]\|x_n-j_n\|+\gamma_n\|x_n-Tx_n\|\Big\}\\ &+\alpha_n\delta\Big\{[1-\beta_n(1-\delta)][1-\gamma_n(1-\delta)]\|x_n-j_n\|+(1-\beta_n)\gamma_n\|x_n-Tx_n\|\Big\}\\ \leq&(1-\alpha_n)\|x_n-Tx_n\|+(1-\alpha_n)\delta[1-\gamma_n(1-\delta)]\|x_n-j_n\|\\ &+(1-\alpha_n)\gamma_n\delta\|x_n-Tx_n\|+\alpha_n\delta[1-\beta_n(1-\delta)][1-\gamma_n(1-\delta)\|x_n-j_n\|\\ &+\alpha_n\gamma_n\delta(1-\beta_n)\|x_n-Tx_n\|]\end{aligned} \tag{31}$$

$$\begin{aligned}\|x_{n+1}-j_{n+1}\| =&\|Tv_n-Tk_n\|\\ \leq&\delta\|v_n-k_n\|\\ \leq&\delta\Big\{\delta[(1-\alpha_n)+\alpha_n[1-\beta_n(1-\delta)]][1-\gamma_n(1-\delta)]\|x_n-j_n\|\\ &+\delta[1-\alpha_n+\gamma_n\delta]\|x_n-Tx_n\|.\Big\}\end{aligned} \tag{32}$$

Since $\delta\in(0,1)$ and $\beta_n\in[0,1]$, then $1-\beta_n(1-\delta)<1$ and $\delta^2<1$, such that (32) becomes

$$\|x_{n+1}-j_{n+1}\|\leq[1-\gamma_n(1-\delta)]\|x_n-j_n\|+\delta[1-\alpha_n+\gamma_n\delta]\|x_n-Tx_n\|.$$

Let $\sigma_n=\|x_n-j_n\|$, $\eta_n=(1-\delta)\gamma_n\in(0,1)$ and $\lambda_n=\delta[1-\alpha_n+\gamma_n\delta]\|x_n-Tx_n\|$.
Since $Tx^*=x^*$ and $\|x_n-x^*\|\to0$ as $n\to\infty$, then

$$\begin{aligned}\|x_n-Tx_n\| =&\|Tx_n-Tx^*+Tx^*-x_n\|\\ \leq&\|Tx_n-Tx^*\|+\|Tx^*-x_n\|\\ \leq&\delta\|x_n-x^*\|+\|x_n-x^*\|\\ =&(1+\delta)\|x_n-x^*\|\end{aligned} \tag{33}$$

It follows $\|x_n-Tx_n\|\to0$, as $n\to\infty$ and $\lambda_n\to0$ as $n\to\infty$.
By Lemma 2.7, we have that $\lim_{n\to\infty}\|x_n-j_n\|=0$.
Since

$$\begin{aligned}\|j_n-x^*\| =&\|j_n-x_n+x_n-x^*\|\\ \leq&\|j_n-x_n\|+\|x_n-x^*\|\\ =&\|x_n-j_n\|+\|x_n-x^*\|\end{aligned} \tag{34}$$

we have that

$$\lim_{n\to\infty}\|j_n-x^*\|=0.$$

Next, we show that (2)⇒ (1)

$$\begin{aligned}\|m_n-t_n\| =&\|(1-\gamma_n)j_n+\gamma_nTj_n-(1-\gamma_n)x_n-\gamma_nTx_n\|\\ \leq&(1-\gamma_n)\|j_n-x_n\|+\gamma_n\|Tj_n-Tx_n\|\\ \leq&(1-\gamma_n)\|j_n-x_n\|+\gamma_n\delta\|j_n-x_n\|\\ =&[1-\gamma_n(1-\delta)]\|j_n-x_n\| \end{aligned}\tag{35}$$

$$\begin{aligned}\|l_n-u_n\| =&\|(1-\beta_n)m_n+\beta_nTm_n-(1-\beta_n)x_n-\beta_nTt_n\|\\ \leq&(1-\beta_n)\|m_n-x_n\|+\beta_n\|Tm_n-Tt_n\|\\ \leq&(1-\beta_n)\|m_n-x_n\|+\beta_n\delta\|m_n-t_n\|\\ \leq&(1-\beta_n)\|m_n-x_n\|+\beta_n\delta[1-\gamma_n(1-\delta)]\|j_n-x_n\| \end{aligned}\tag{36}$$

$$\begin{aligned}\|m_n-x_n\| =&\|(1-\gamma_n)j_n+\gamma_nTj_n-x_n\|\\ \leq&(1-\gamma_n)\|j_n-x_n\|+\gamma_n\|Tj_n-x_n\|\\ =&(1-\gamma_n)\|j_n-x_n\|+\gamma_n\|Tj_n-Tx_n+Tx_n-x_n\|\\ \leq&(1-\gamma_n)\|j_n-x_n\|+\gamma_n\delta\|j_n-x_n\|+\gamma_n\|Tx_n-x_n\|\\ =&[1-\gamma_n(1-\delta)]\|j_n-x_n\|+\gamma_n\|Tx_n-x_n\| \end{aligned}\tag{37}$$

Putting (37) in (36)

$$\begin{aligned}\|l_n-u_n\| \leq&(1-\beta_n)\{[1-\gamma_n(1-\delta)]\|j_n-x_n\|+\gamma_n\|Tx_n-x_n\|\}+\\ &\beta_n\delta[1-\gamma_n(1-\delta)]\|j_n-x_n\|\\ =&[1-\beta_n(1-\delta)][1-\gamma_n(1-\delta)]\|j_n-x_n\|+(1-\beta_n)\gamma_n\|Tx_n-x_n\| \end{aligned}\tag{38}$$

$$\begin{aligned}\|k_n-v_n\| =&\|(1-\alpha_n)Tm_n+\alpha_nTl_n-(1-\alpha_n)x_n-\alpha_nTu_n\|\\ \leq&(1-\alpha_n)\|Tm_n-x_n\|+\alpha_n\|Tl_n-Tu_n\|\\ \leq&(1-\alpha_n)\|Tm_n-Tx_n+Tx_n-x_n\|+\alpha_n\delta\|l_n-u_n\|\\ \leq&(1-\alpha_n)\delta\|m_n-x_n\|+(1-\alpha_n)\|Tx_n-x_n\|+\alpha_n\delta\|l_n-u_n\|. \end{aligned}\tag{39}$$

Putting (37) and (38) in (39)

$$\begin{aligned}\|k_n-v_n\| \leq&(1-\alpha_n)\delta\Big\{[1-\gamma_n(1-\delta)]\|j_n-x_n\|+\gamma_n\|Tx_n-x_n\|\Big\}\\ &+(1-\alpha_n)\|Tx_n-x_n\|+\alpha_n\delta\Big\{[1-\beta_n(1-\delta)][1-\gamma_n(1-\delta)]\|j_n-x_n\|\\ &+(1-\beta_n)\gamma_n\|Tx_n-x_n\|\Big\} \end{aligned}\tag{40}$$

$$\begin{aligned}\|j_{n+1}-x_{n+1}\| =&\|Tk_n-Tv_n\|\\ \leq&\delta\|k_n-v_n\|\\ \leq&\delta^2[(1-\alpha_n)+\alpha_n[1-\beta_n(1-\delta)]][1-\gamma_n(1-\delta)]\|j_n-x_n\|\\ &+\Big\{(1-\alpha_n)\gamma_n\delta^2+(1-\alpha_n)\delta+(1-\beta_n)\alpha_n\gamma_n\delta^2\Big\}\|Tx_n-x_n\|.\end{aligned} \tag{41}$$

Since $\delta\in(0,1)$ and $\alpha_n,\beta_n\in[0,1]$ for each $n\in\mathbb{N}$, $\delta^2<1$ and $1-\beta_n(1-\delta)<1$ such that (41) reduces to

$$\|j_{n+1}-x_{n+1}\|\leq[1-\gamma_n(1-\delta)]\|j_n-x_n\|+\Big\{(1-\alpha_n)\gamma_n\delta^2+(1-\alpha_n)\delta+(1-\beta_n)\alpha_n\gamma_n\delta^2\Big\}\|Tx_n-x_n\|.$$

Let $\sigma_n=\|j_n-x_n\|$, $\eta_n=(1-\delta)\gamma_n\in(0,1)$ and $\lambda_n=[(1-\alpha_n)\gamma_n\delta^2+(1-\alpha_n)\delta+(1-\beta_n)\alpha_n\gamma_n\delta^2]\|Tx_n-x_n\|$.
Since $Tx^*=x^*$ and $\|x_n-x^*\|\to 0$ as $n\to\infty$, then

$$\begin{aligned}\|Tx_n-x_n\| =&\|Tx_n-Tx^*+Tx^*-x_n\|\\ \leq&\delta\|x_n-x^*\|+\|x_n-x^*\|\\ =&(1+\delta)\|x_n-x^*\|\end{aligned}$$

so that $\|Tx_n-x_n\|\leq(1+\delta)\|x_n-x^*\|\to 0$, as $n\to\infty$.
This means that $\lambda_n\to 0$ as $n\to\infty$. By Lemma 2.7, we have that $\lim\limits_{n\to\infty}\|j_n-x_n\|=0$.
Furthermore, since

$$\begin{aligned}\|x_n-x^*\| =&\|x_n-j_n+j_n-x^*\|\\ \leq&\|x_n-j_n\|+\|j_n-x^*\|\\ =&\|j_n-x_n\|+\|j_n-x^*\|\to 0 \text{ as } n\to\infty,\end{aligned}$$

thus $\lim\limits_{n\to\infty}\|x_n-x^*\|=0$. Hence, the proof is complete. □

4. Data Dependence and Stability Results

In this section, data dependence result for the new iteration process and thereafter, stability result will be proved.

Theorem 4.1. Let $\tilde{T}$ be an approximate mapping of T satisfying the contraction mapping condition (15). Let $\{x_n\}_{n=1}^{\infty}$ be an iterative sequence generated by the Picard-Noor hybrid iteration process (14) for

T and define an iterative sequence $\{\tilde{x}_n\}_{n=1}^{\infty}$ as follows;

$$\begin{cases} \tilde{x}_0 = \tilde{x} \in C \\ \tilde{x}_{n+1} = \tilde{T}\tilde{v}_n \\ \tilde{v}_n = (1-\alpha_n)\tilde{x}_n + \alpha_n \tilde{T}\tilde{u}_n \\ \tilde{u}_n = (1-\beta_n)\tilde{x}_n + \beta_n \tilde{T}\tilde{t}_n \\ \tilde{t}_n = (1-\gamma_n)\tilde{x}_n + \gamma_n \tilde{T}\tilde{x}_n,\ n \in \mathbb{N} \end{cases} \tag{42}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\} \in (0,1)$ satisfying the following conditions;

(a) $\frac{1}{2} \leq \alpha_n$ for all $n \in \mathbb{N}$, and

(b) $\sum_{n=0}^{\infty} \alpha_n = \infty$.

If $Tx^* = x^*$ and $\tilde{T}\tilde{x}^* = \tilde{x}^*$ are such that $\tilde{x}_n \longrightarrow \tilde{x}^*$, then we have $\|x^* - \tilde{x}^*\| \leq \frac{7\varepsilon}{1-\delta}$, where $\varepsilon > 0$ is a fixed constant.

Proof. Using (15), (14) and (42), we have

$$\begin{aligned} \|t_n - \tilde{t}_n\| =& \|(1-\gamma_n)x_n + \gamma_n T x_n - (1-\gamma_n)\tilde{x}_n - \gamma_n \tilde{T}\tilde{x}_n\| \\ \leq& (1-\gamma_n)\|x_n - \tilde{x}_n\| + \gamma_n \|T x_n - \tilde{T}\tilde{x}_n\| \\ =& (1-\gamma_n)\|x_n - \tilde{x}_n\| + \gamma_n \|T x_n - T\tilde{x}_n + T\tilde{x}_n - \tilde{T}\tilde{x}_n\| \\ \leq& (1-\gamma_n)\|x_n - \tilde{x}_n\| + \gamma_n \|T x_n - T\tilde{x}_n\| + \gamma_n \varepsilon \\ \leq& (1-\gamma_n)\|x_n - \tilde{x}_n\| + \gamma_n \delta \|x_n - \tilde{x}_n\| + \gamma_n \varepsilon \\ =& [1-(1-\delta)\gamma_n]\|x_n - \tilde{x}_n\| + \gamma_n \varepsilon \end{aligned} \tag{43}$$

$$\begin{aligned} \|u_n - \tilde{u}_n\| =& \|(1-\beta_n)x_n + \beta_n T t_n - (1-\beta_n)\tilde{x}_n - \beta_n \tilde{T}\tilde{t}_n\| \\ \leq& (1-\beta_n)\|x_n - \tilde{x}_n\| + \beta_n \|T t_n - \tilde{T}\tilde{t}_n\| \\ =& (1-\beta_n)\|x_n - \tilde{x}_n\| + \beta_n \|T t_n - T\tilde{t}_n + T\tilde{t}_n - \tilde{T}\tilde{t}_n\| \\ \leq& (1-\beta_n)\|x_n - \tilde{x}_n\| + \beta_n \|T t_n - T\tilde{t}_n\| + \beta_n \varepsilon \\ \leq& (1-\beta_n)\|x_n - \tilde{x}_n\| + \beta_n \delta \|t_n - \tilde{t}_n\| + \beta_n \varepsilon, \end{aligned} \tag{44}$$

combining (43) and (44)

$$\begin{aligned} \|u_n - \tilde{u}_n\| \leq& (1-\beta_n)\|x_n - \tilde{x}_n\| + \beta_n \delta \{[1-(1-\delta)\gamma_n]\|x_n - \tilde{x}_n\| + \gamma_n \varepsilon\} \\ &+ \beta_n \varepsilon \\ =& \{(1-\beta_n) + \beta_n \delta [1-(1-\delta)\gamma_n]\}\|x_n - \tilde{x}_n\| + \beta_n \gamma_n \delta \varepsilon + \beta_n \varepsilon \\ =& [1-\beta_n(1-\delta[1-(1-\delta)\gamma_n])]\|x_n - \tilde{x}_n\| + \beta_n \gamma_n \delta \varepsilon + \beta_n \varepsilon \end{aligned} \tag{45}$$

$$\begin{aligned}\|v_n - \tilde{v}_n\| =& \|(1-\alpha_n)x_n + \alpha_n T u_n - (1-\alpha_n)\tilde{x}_n - \alpha_n \tilde{T}\tilde{u}_n\| \\ \le& (1-\alpha_n)\|x_n - \tilde{x}_n\| + \alpha_n \|T u_n - \tilde{T}\tilde{u}_n\| \\ =& (1-\alpha_n)\|x_n - \tilde{x}_n\| + \alpha_n \|T u_n - T\tilde{u}_n + T\tilde{u}_n - \tilde{T}\tilde{u}_n\| \\ \le& (1-\alpha_n)\|x_n - \tilde{x}_n\| + \alpha_n \delta \|u_n - \tilde{u}_n\| + \alpha_n \varepsilon, \end{aligned} \tag{46}$$

puting (45) in (46), we have;

$$\begin{aligned}\|v_n - \tilde{v}_n\| \le& (1-\alpha_n)\|x_n - \tilde{x}_n\| \\ &+ \alpha_n \delta \{[1-\beta_n(1-\delta[1-(1-\delta)\gamma_n])]\|x_n - \tilde{x}\| + \beta_n \gamma_n \delta \varepsilon\} + \alpha_n \varepsilon \\ \le& (1-\alpha_n)\|x_n - \tilde{x}_n\| + \alpha_n \delta [1-\beta_n(1-\delta[1-(1-\delta)\gamma_n])]\|x_n - \tilde{x}_n\| \\ &+ \alpha_n \beta_n \gamma_n \delta \varepsilon + \alpha_n \beta_n \delta \varepsilon + \alpha_n \varepsilon \\ =& \{(1-\alpha_n) + \alpha_n \delta [1-\beta_n(1-\delta[1-(1-\delta)\gamma_n])]\}\|x_n - \tilde{x}_n\| \\ &+ \alpha_n \beta_n \gamma_n \delta \varepsilon + \alpha_n \beta_n \delta \varepsilon + \alpha_n \varepsilon \\ =& [1-\alpha_n(1-\delta[1-\beta_n(1-\delta[1-\gamma_n(1-\delta)])])]\|x_n - \tilde{x}_n\| \\ &+ \alpha_n \beta_n \gamma_n \delta \varepsilon + \alpha_n \beta_n \delta \varepsilon + \alpha_n \varepsilon. \end{aligned} \tag{47}$$

Finally,

$$\begin{aligned}\|x_{n+1} - \tilde{x}_{n+1}\| =& \|T v_n - \tilde{T}\tilde{v}_n\| \\ =& \|T v_n - T\tilde{v}_n + T\tilde{v}_n - \tilde{T}\tilde{v}_n\| \\ \le& \delta \|v_n - \tilde{v}_n\| + \varepsilon, \end{aligned} \tag{48}$$

putting (47) in (48), we have

$$\begin{aligned}\|x_{n+1} - \tilde{x}_{n+1}\| \le& \delta[1-\alpha_n(1-\delta[1-\beta_n(1-\delta[1-\gamma_n(1-\delta)])])]\|x_n - \tilde{x}_n\| \\ &+ \alpha_n \beta_n \gamma_n \delta^2 \varepsilon + \alpha_n \beta_n \delta^2 \varepsilon + \alpha_n \delta \varepsilon + \varepsilon. \end{aligned}$$

Since $0 < \delta < 1$, $\alpha_n, \beta_n, \gamma_n \in (0,1)$, $n \in \mathbb{N}$, then

$$\begin{aligned} &\delta < 1 \\ &\alpha_n \beta_n \gamma_n \delta^2 < 1 \\ &\alpha_n \beta_n \delta^2 < 1 \\ &[1-\beta_n(1-\delta[1-\gamma_n(1-\delta)])] < 1 \end{aligned}$$

and recalling hypothesis (a) of Theorem 4.1 where $\frac{1}{2} \le \alpha_n$, we have;

$$1 - \alpha_n \le \alpha_n,$$

this implies

$$1 = 1 - \alpha_n + \alpha_n \le \alpha_n + \alpha_n = 2\alpha_n.$$

Therefore, we have that

$$\begin{aligned}\|x_{n+1}-\tilde{x}_{n+1}\| \leq&[1-\alpha_n(1-\delta)]\|x_n-\tilde{x}_n\|+\alpha_n\varepsilon+3\varepsilon\\ =&[1-\alpha_n(1-\delta)]\|x_n-\tilde{x}_n\|+\alpha_n\varepsilon+3(1-\alpha_n+\alpha_n)\varepsilon\\ \leq&[1-\alpha_n(1-\delta)]\|x_n-\tilde{x}_n\|+\alpha_n(1-\delta)\frac{7\varepsilon}{(1-\delta)}.\end{aligned}$$

Let $\rho_n := \|x_n-\tilde{x}_n\|$, $\zeta_n := \alpha_n(1-\delta)\in(0,1)$, $\eta_n := \frac{7\varepsilon}{1-\delta}$.
From Lemma 2.8, it follows that

$$0\leq \limsup_{n\to\infty}\|x_n-\tilde{x}_n\|\leq \limsup_{n\to\infty}\frac{7\varepsilon}{1-\delta}.$$

From Theorem 3.1, it follows that $\lim_{n\to\infty} x_n = x^*$. Using condition (b) of Theorem 4.1, i.e. $\lim_{n\to\infty}\tilde{x}_n=\tilde{x^*}$, then we have

$$\|x^*-\tilde{x^*}\|\leq\frac{7\varepsilon}{1-\delta}.$$

Hence, the proof is complete. □

Theorem 4.2. Let $\mathscr{X}$ be a Banach space and $T: C\to C$ be a contraction mapping satisfying condition with $0\leq\delta<1$. Suppose that T has a fixed point $x^*\in\mathfrak{F}(T)\neq\emptyset$. Let $\{x_n\}_{n=0}^{\infty}$ be a sequence defined by the Picard-Noor hybrid iteration process (14) satisfying $\sum_{n=0}^{\infty}\alpha_n=\infty$, $n\in\mathbb{N}$ and T converges to a fixed point x^*. Then (14) is stable with respect to T.

Proof. Assume that $\{r_n\}_{n=0}^{\infty}\subset\mathscr{X}$ is an arbitrary sequence in C and assume that the sequence generated by the Picard-Noor hybrid iteration process (14) is $x_{n+1}=f(T,x_n)$ converging to a unique fixed point x^*. Let $\varepsilon_n=\|r_{n+1}-f(T,r_n)\|$. It is our aim to show that $\lim_{n\to\infty}\varepsilon_n=0$ if and only if $\lim_{n\to\infty}\|r_n-x^*\|=0$.
Assume that $\lim_{n\to\infty}\varepsilon_n=0$. From elementary analysis, we have

$$\begin{aligned}\|r_{n+1}-x^*\| =&\|r_{n+1}-f(T,r_n)+f(T,r_n)-x^*\|\\ \leq&\|r_{n+1}-f(T,r_n)\|+\|f(T,r_n)-x^*\|\\ \leq&\varepsilon_n+\|Tv_n-x^*\|\\ \leq&\varepsilon_n+\delta\|v_n-x^*\| \qquad (49)\end{aligned}$$

$$\begin{aligned}\|v_n-x^*\| =&\|(1-\alpha_n)r_n+\alpha_nTu_n-x^*\|\\ \leq&(1-\alpha_n)\|r_n-x^*\|+\alpha_n\|Tu_n-x^*\|\\ \leq&(1-\alpha_n)\|r_n-x^*\|+\alpha_n\delta\|u_n-x^*\| \qquad (50)\end{aligned}$$

$$\begin{aligned}\|u_n-x^*\| =&\|(1-\beta_n)r_n+\beta_nTt_n-x^*\|\\ \leq&(1-\beta_n)\|r_n-x^*\|+\beta_n\delta\|t_n-x^*\| \qquad (51)\\ \|t_n-x^*\| =&\|(1-\gamma_n)r_n+\gamma_nTr_n-x^*\|\\ \leq&(1-\gamma_n)\|r_n-x^*\|+\gamma_n\|Tr_n-x^*\|\\ \leq&(1-\gamma_n)\|r_n-x^*\|+\gamma_n\delta\|r_n-x^*\|\\ =&[(1-\gamma_n)+\gamma_n\delta]\|r_n-x^*\|\\ =&[1-\gamma_n(1-\delta)]\|r_n-x^*\|. \qquad (52)\end{aligned}$$

Combining (49)-(52), we have

$\|r_{n+1}-x^*\|\leq\varepsilon_n+\delta(1-\alpha_n(1-\delta[1-\beta_n(1-\delta[1-\gamma_n(1-\delta)])]))\|r_n-x^*\|.$

Since $0<\delta<1$ and $\{\alpha_n\},\{\beta_n\},\{\gamma_n\}\in[0,1]$, then we have that

$$\delta(1-\alpha_n(1-\delta[1-\beta_n(1-\delta[1-\gamma_n(1-\delta)])]))<1,$$

thereby giving rise to

$$\|r_{n+1}-x^*\|\leq\varepsilon_n+\|r_n-x^*\|.$$

By Lemma 2.5, we have $\lim_{n\to\infty}\|r_n-x^*\|=0$, that is $r_n\to x^*$ as $n\to\infty$.

Conversely, assume that $\lim_{n\to\infty}r_n=x^*$, then

$$\begin{aligned}\varepsilon_n =&\|r_{n+1}-f(T,r_n)\|\\ =&\|r_{n+1}-x^*+x^*-f(T,r_n)\|\\ \leq&\|r_{n+1}-x^*\|+\|x^*-Tv_n\|\\ \leq&\|r_{n+1}-x^*\|+\delta\|v_n-x^*\|\\ \leq&\|r_{n+1}-x^*\|+\delta\{(1-\alpha_n)\|r_n-x^*\|+\alpha_n\delta\|u_n-x^*\|\}\\ =&\|r_{n+1}-x^*\|+\delta(1-\alpha_n)\|r_n-x^*\|+\alpha_n\delta^2\|u_n-x^*\|\\ \leq&\|r_{n+1}-x^*\|+\delta(1-\alpha_n)\|r_n-x^*\|+\alpha_n\delta^2\{(1-\beta_n)\|r_n-x^*\|+\beta_n\delta\|t_n-x^*\|\}\\ =&\|r_{n+1}-x^*\|+\delta(1-\alpha_n)\|r_n-x^*\|+\alpha_n\delta^2(1-\beta_n)\|r_n-x^*\|\\ &+\alpha_n\beta_n\delta^3\|t_n-x^*\|\\ =&\|r_{n+1}-x^*\|+\{\delta(1-\alpha_n)+\alpha_n\delta^2(1-\beta_n)\}\|r_n-x^*\|\\ &+\alpha_n\beta_n\delta^3\|t_n-x^*\|\\ \leq&\|r_{n+1}-x^*\|+\{\delta(1-\alpha_n)+\alpha_n\delta^2(1-\beta_n)\}\|r_n-x^*\|\\ &+\alpha_n\beta_n\delta^3\{[1-\gamma_n(1-\delta)]\|r_n-x^*\|\}\\ =&\|r_{n+1}-x^*\|+\{\delta(1-\alpha_n)+\alpha_n\delta^2(1-\beta_n)+\alpha_n\beta_n\delta^3[1-\gamma_n(1-\delta)]\}\|r_n-x^*\|.\end{aligned}$$

Since, $\lim_{n\to\infty}r_n=x^*$, it follows that $\lim_{n\to\infty}\varepsilon_n=0$. Hence completing the proof. □

5. Application to Delay Differential Equations

In this section, we consider the application of the Picard-Noor hybrid iteration process in obtaining the solution of delay differential equations. Here, we consider the following delay differential equation;

$$x^{'}(t) = f(t,(t),x(t-h)), \quad t \in [t_0,b] \tag{53}$$

with initial condition

$$x(t) = \varphi(t), \qquad t \in [t_0-h,t_0]. \tag{54}$$

It is our aim to show that the sequence generated by the Picard-Noor iteration process (14) converges to the solution of (53)-(54).
Let $C([a,b])$ denote the space of continuous functions on a closed interval $[a,b]$ and $\|\cdot\|_\infty$, a Chebyshev norm $\|x-y\|_\infty = \max\limits_{t\in[a,b]} |x(t)-y(t)|$.
It is obvious that $(C([a,b]),\|\cdot\|_\infty)$ is a Banach space.
Suppose that the following conditions are satisfied;

(D_1): $t_0,\ b \in \mathbb{R}, b>0$;
(D_2): $f \in C([a,b]\times\mathbb{R}^2,\mathbb{R})$;
(D_3): $\varphi(t) \in C([t_0-h,b],\mathbb{R})$;
(D_4): there exists $L_f > 0$ such that

$$|f(t,u_,u_2)-f(t,v_1,v_2)| \leq L_f\sum_{i=1}^{2}|u_i-v_i|,\ \forall u_i,v_i\in\mathbb{R},\ i=1,2,\ t\in[t_0,b]$$

(D_5): $2L_f(b-t_0)<1$.

If the solution of (53)-(54) exists, it will take the form of the following integral equation;

$$x(t) = \begin{cases} \varphi(t), & t\in[t_0-h,t_0] \\ \varphi(t_0)+\int_{t_0}^{t} f(s,x(s),x(s-h))ds, & t\in[t_0,b] \end{cases}$$

where $x \in C([t_0-h,b],\mathbb{R})\cap C^1([t_0,b],\mathbb{R})$.
The following result was established by Coman et al [8];

Theorem 5.1. Assume that conditions (D_1)–(D_5) are satisfied. Then problem (53)-(54) has a unique solution, say $x^* \in C([t_0-h,b],\mathbb{R})\cap C^1([t_0,b],\mathbb{R})$ and $x^* = \lim\limits_{n\to\infty} T^n(x)$ for any $x\in C([t_0-h,b],\mathbb{R})$.

We now give the following result for the solution of delay differential equation utilizing our iteration process (14).

Theorem 5.2. Suppose that conditions (D_1)–(D_5) are satisfied. Assume that problem (53)–(54) has a unique solution $x^* \in C([t_0-h,b],\mathbb{R})\cap C^1([t_0,b],\mathbb{R})$ and the sequence $\{x_n\}_{n=0}^{\infty}$ generated by the Picard-Noor hybrid iteration process (14) for $\{\alpha_n\},\{\beta_n\},\{\gamma_n\} \in (0,1)$ such that $\sum_{n=0}^{\infty} \alpha_n = \infty$, converges to x^*.

Proof. Let $\{x_n\}$ be an iterative sequence generated by (14) for the following operator;

$$Tx(t) = \begin{cases} \varphi(t), & t \in [t_0 - h, t_0] \\ \varphi(t_0) + \int_{t_0}^{t} f(s,x(s),x(s-h))ds, & t \in [t_0,b]. \end{cases} \tag{55}$$

Let $\alpha_n, \beta_n, \gamma_n \in (0,1)$ for all $n \in \mathbb{N}$, such that $\sum_{n=0}^{\infty} \alpha_n = \infty$. Let $x^* \in \mathfrak{F}(T) \neq \emptyset$ be the fixed point of T. We are to show that $x_n \to x^*$ as $n \to \infty$.
For each $t \in [t_0 - h, t_0]$, it is clear to see that $x_n \to x^*$ as $n \to \infty$ and for $t \in [t_0, b]$, we have,

$$\begin{aligned}
\|t_n - x^*\|_\infty =& \|(1-\gamma_n)x_n + \gamma_n T x_n - x^*\|_\infty \\
\leq& (1-\gamma_n)\|x_n - x^*\|_\infty + \gamma_n \|Tx_n - x^*\|_\infty \\
=& (1-\gamma_n)\|x_n - x^*\|_\infty + \gamma_n \|Tx_n - Tx^*\|_\infty \\
=& (1-\gamma_n)\|x_n - x^*\|_\infty + \gamma_n \max_{t\in[t_0-h,b]} |Tx_n(s) - Tx^*(s)| \\
=& (1-\gamma_n)\|x_n - x^*\|_\infty + \gamma_n \max_{t\in[t_0-h,b]} \Big|\varphi(t_0) \\
&+ \int_{t_0}^{t} f(s,x_n(s),x_n(s-h))ds - \varphi(t_0) - \int_{t_0}^{t} f(s,x^*(s),x^*(s-h))ds\Big| \\
=& (1-\gamma_n)\|x_n - x^*\|_\infty + \gamma_n \max_{t\in[t_0-h,b]} \Big| \int_{t_0}^{t} f(s,x_n(s),x_n(s-h))ds \\
&- \int_{t_0}^{t} f(s,x^*(s),x^*(s-h))ds\Big| \\
\leq& (1-\gamma_n)\|x_n - x^*\|_\infty + \gamma_n \max_{t\in[t_0-h,b]} \int_{t_0}^{t} \Big| f(s,x_n(s),x_n(s-h)) \\
&- f(s,x^*(s),x^*(s-h))\Big| ds
\end{aligned}$$

$$
\begin{aligned}
&\leq (1-\gamma_n)\|x_n - x^*\|_\infty + \gamma_n \max_{t\in[t_0-h,b]} \int_{t_0}^{t} L_f(|x_n(s)-x^*(s)| \\
&\quad + |x_n(s-h)-x^*(s-h)|)ds \\
&\leq (1-\gamma_n)\|x_n - x^*\|_\infty + \gamma_n \int_{t_0}^{t} L_f\Big(\max_{t\in[t_0-h,b]}|x_n(s)-x^*(s)| \\
&\quad + \max_{t\in[t_0-h,b]}|x_n(s-h)-x^*(s-h)|\Big)ds \qquad (56)\\
&\leq (1-\gamma_n)\|x_n - x^*\|_\infty + \gamma_n \int_{t_0}^{t} L_f(\|x_n - x^*\|_\infty + \|x_n - x^*\|_\infty)ds \\
&\leq (1-\gamma_n)\|x_n - x^*\|_\infty + 2\gamma_n L_f(t-t_0)\|x_n - x^*\|_\infty \\
&= [(1-\gamma_n) + 2\gamma_n L_f(b-t_0)]\|x_n - x^*\|_\infty \\
&= [1-\gamma_n(1-2L_f(b-t_0))]\|x_n - x^*\|_\infty
\end{aligned}
$$

$$
\begin{aligned}
\|u_n - x^*\| &= \|(1-\beta_n)x_n + \beta_n T t_n - x^*\|_\infty \\
&\leq (1-\beta_n)\|x_n - x^*\|_\infty + \beta_n\|T t_n - x^*\|_\infty \\
&= (1-\beta_n)\|x_n - x^*\|_\infty + \beta_n\|T t_n - T x^*\|_\infty \\
&= (1-\beta_n)\|x_n - x^*\|_\infty + \beta_n \max_{t\in[t_0-h,b]} |T t_n(t) - T x^*(t)| \\
&= (1-\beta_n)\|x_n - x^*\|_\infty + \beta_n \max_{t\in[t_0-h,b]} |\varphi(t_0) \\
&\quad + \int_{t_0}^{t} f(s, t_n(s), t_n(s-h))ds - \varphi(t_0) - \int_{t_0}^{t} f(s, x^*(s), x^*(s-h))ds| \\
&\leq (1-\beta_n)\|x_n - x^*\|_\infty + \beta_n \max_{t\in[t_0-h,b]} \int_{t_0}^{t} |f(s, t_n(s), t_n(s-h)) \\
&\quad - f(s, x^*(s), x^*(s-h))|ds \\
&\leq (1-\beta_n)\|x_n - x^*\|_\infty + \beta_n \max_{t\in[t_0-h,b]} \int_{t_0}^{t} L_f(|t_n(s)-x^*(s)| \\
&\quad + |t_n(s-h)-x^*(s-h)|)ds \\
&\leq (1-\beta_n)\|x_n - x^*\|_\infty + \beta_n \int_{t_0}^{t} L_f(\max_{t\in[t_0-h,b]}|t_n(s)-x^*(s)| \\
&\quad + \max_{t\in[t_0-h,b]}|t_n(s-h)-x^*(s-h)|)ds \\
&\leq (1-\beta_n)\|x_n - x^*\|_\infty + \beta_n \int_{t_0}^{t} L_f(\|t_n - x^*\|_\infty + \|t_n - x^*\|_\infty)ds \\
&\leq (1-\beta_n)\|x_n - x^*\|_\infty + 2\beta_n L_f(t-t_0)\|t_n - x^*\|_\infty \\
&\leq (1-\beta_n)\|x_n - x^*\|_\infty + 2\beta_n L_f(t-t_0)[1-\gamma_n(1-2L_f(b-t_0))]\|x_n - x^*\|_\infty \qquad (57)
\end{aligned}
$$

by condition (D_5), that is $2L_f(t-t_0)<1$, we have;

$$\begin{aligned}\|u_n-x^*\|_\infty \leq&(1-\beta_n)\|x_n-x^*\|_\infty+\beta_n[1-\gamma_n(1-2L_f(b-t_0))]\|x_n-x^*\|_\infty\\ =&\{(1-\beta_n)+\beta_n[1-\gamma_n(1-2L_f(b-t_0))]\}\|x_n-x^*\|_\infty\\ \leq&[1-\beta_n(1-[1-\gamma_n(1-2L_f(b-t_0))])]\|x_n-x^*\|_\infty\end{aligned} \tag{58}$$

$$\begin{aligned}\|v_n-x^*\|_\infty =&\|(1-\alpha_n)x_n+\alpha_nTu_n-x^*\|_\infty\\ \leq&(1-\alpha_n)\|x_n-x^*\|_\infty+\alpha_n\|Tu_n-x^*\|_\infty\\ =&(1-\alpha_n)\|x_n-x^*\|_\infty+\alpha_n\|Tu_n-Tx^*\|_\infty\\ =&(1-\alpha_n)\|x_n-x^*\|_\infty+\alpha_n\max_{t\in[t_0-h,b]}|Tu_n(t)-Tx^*(t)|\\ =&(1-\alpha_n)\|x_n-x^*\|_\infty+\alpha_n\max_{t\in[t_0-h,b]}|\varphi(t_0)+\int_{t_0}^t f(s,u_n(s),u_n(s-h))ds\\ &-\varphi(t_0)-\int_{t_0}^t f(s,x^*(s),x^*(s-h))ds|\\ \leq&(1-\alpha_n)\|x_n-x^*\|_\infty+\alpha_n\max_{t\in[t_0-h,b]}\int_{t_0}^t|f(s,u_n(s),u_n(s-h))\\ &-f(s,x^*(s),x^*(s-h))|ds\\ \leq&(1-\alpha_n)\|x_n-x^*\|_\infty+\alpha_n\max_{t\in[t_0-h,b]}\int_{t_0}^t L_f(|u_n(s)-x^*(s)|\\ &+|u_n(s-h)-x^*(s-h)|)ds\\ \leq&(1-\alpha_n)\|x_n-x^*\|_\infty+\alpha_n\int_{t_0}^t L_f(\max_{t\in[t_0-h,b]}|u_n(s)-x^*(s)|\\ &+\max_{t\in[t_0-h,b]}|u_n(s-h)-x^*(s-h)|)ds\\ \leq&(1-\alpha_n)\|x_n-x^*\|_\infty+\alpha_n\int_{t_0}^t L_f(\|u_n-x^*\|_\infty+\|u_n-x^*\|_\infty)ds\\ \leq&(1-\alpha_n)\|x_n-x^*\|_\infty+2\alpha_nL_f(b-t_0)\|u_n-x^*\|_\infty\\ \leq&(1-\alpha_n)\|x_n-x^*\|_\infty\\ &+2\alpha_nL_f(b-t_0)[1-\beta_n(1-[1-\gamma_n(1-2L_f(b-t_0))])]\|x_n-x^*\|_\infty\\ \leq&[1-\alpha_n[1-\beta_n(1-[1-\gamma_n(1-2L_f(b-t_0))])]]\|x_n-x^*\|_\infty\end{aligned} \tag{59}$$

Finally,

$$
\begin{aligned}
\|x_{n+1}-x^*\|_\infty =& \|Tv_n - x^*\|_\infty \\
=& \|Tv_n - Tx^*\|_\infty \\
=& \max_{t\in[t_0-h,b]} |Tv_n(t) - Tx^*(t)| \\
=& \max_{t\in[t_0-h,b]} |\varphi(t_0) + \int_{t_0}^{t} f(s, v_n(s), v_n(s-h))ds - \varphi(t_0) \\
& - \int_{t_0}^{t} f(s, x^*(s), x^*(s-h))ds|
\end{aligned}
$$

$$
\begin{aligned}
&= \max_{t\in[t_0-h,b]} |\int_{t_0}^{t} f(s, v_n(s), v_n(s-h))ds - \int_{t_0}^{t} f(s, x^*(s), x^*(s-h))ds| \\
&\leq \max_{t\in[t_0-h,b]} \int_{t_0}^{t} |f(s, v_n(s), v_n(s-h)) - f(s, x^*(s), x^*(s-h))|ds \\
&\leq \max_{t\in[t_0-h,b]} \int_{t_0}^{t} L_f(|v_n(s) - x^*(s)| + |v_n(s-h) - x^*(s-h)|)ds \\
&\leq \int_{t_0}^{t} L_f(\max_{t\in[t_0-h,b]} |v_n(s) - x^*(s)| + \max_{t\in[t_0-h,b]} |v_n(s-h) - x^*(s-h)|)ds \\
&\leq \int_{t_0}^{t} L_f(\|v_n - x^*\|_\infty + \|v_n - x^*\|_\infty)ds \\
&\leq 2L_f(b-t_0)\|v_n - x^*\|_\infty
\end{aligned} \tag{60}
$$

Putting (59) in (60), we have

$$\|x_{n+1}-x^*\|_\infty \leq 2L_f(b-t_0)[1-\alpha_n[1-\beta_n(1-[1-\gamma_n(1-2L_f(b-t_0))])]]\|x_n-x^*\|_\infty$$

Applying condition (D_5), we have

$$\|x_{n+1}-x^*\|_\infty \leq [1-\alpha_n[1-\beta_n(1-[1-\gamma_n(1-2L_f(b-t_0))])]]\|x_n-x^*\|_\infty$$

Let $\mu_n := \alpha_n[1-\beta_n(1-[1-\gamma_n(1-2L_f(b-t_0))])] < 1$ and $s_n := \|x_n - x^*\|_\infty$. By Lemma 2.6, we assume that conditions if μ_n and s_n are satisfied then $\lim_{n\to\infty} \|x_n - x^*\|_\infty = 0$. Hence completing the proof. □

6. Conclusion

We have been able to construct a hybrid four-step iteration process that contains both Picard and Noor iteration processes and show through numerical examples of Example 1 and 2 that our new process converges faster than all of Picard, Mann, Ishikawa, Krasnoselskii, Noor, Picard-Mann, Picard-Krasnoselskii and other iteration processes in literature.

The efficiency of the Picard-Noor iteration process is visualized in Table 1 to Table 4; and graphs in figure 1 and figure 2. Data dependence and stability results was proved for the iteration process. Finally, the proposed iteration process was applied in solving certain delay differential equations. Considering all the results proved in this paper, it is evident that the Picard-Noor hybrid iteration process is a generalization of the results of Ishikawa [11], Krasnoselskii [13], Khan [14], Mann [15], Noor [17], Okeke and Abbas [18] and Okeke [19].

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References

[1] M. Abbas and T. Nazir, *Some new faster iteration process applied to constrained minimization and feasibility problems*, Matematički Vesnik, **66**:2, 223–234, 2014.

[2] R. P. Agarwal, D. O'Regan and D. R. Sahu, *Iterative construction of fixed points of nearly asymptotically nonexpansive mappings*, J. Nonlinear Convex Anal. 8:1, 61–79, 2007.

[3] V. Berinde, *Iterative Approximation of Fixed Points*. Efemeride, Baia Mare, 2002.

[4] V. Berinde, *On the convergence of the Ishikawa iteration in the class of quasi contractive operators.*, Acta Mathematica Universitatis Comenianae, **73**:1, 119–126, 2004.

[5] V. Berinde, *Iterative Approximation of Fixed Points*, Editura Efemeride, Baia Mare, 2002.

[6] V. Berinde, *Picard iteration converges faster than Mann iteration for a class of quasi-contractive operator*, Fixed Point Theory and Applications, **2004**:2, 97–105, http://dx.doi.org/10.1155/s1687182004311058, 2004.

[7] V. Berinde, *Iterative approximation of fixed points*, Springer **1912**, 2007.

[8] G. H. Coman, G. Pavel, I. Rus and I. A. Rus, *Introduction in the theory of operational equations*, Ed. Dacia, Cluj-Napoca, 1976.

[9] C. Garodia and I. Uddin, *A new fixed point algorithm for finding the solution of a delay differential equation*, Mathematics, **5**:4, 3182-3200, hhttps://doi.org/10.3934/math.2020205, 2020.

[10] A. Genel and J. Lindenstrauss, *An example concerning fixed points*, Israel Journal of Mathematics, **22**, 81–86, 1975.

[11] S. Ishikawa, *Fixed points by a new iteration method*, Proceeding of the American Mathematical Society, **44:**1, 147–150, 1974.

[12] J. Jia, K. Shabbir, K. Ahmad, N.A. Shah and T. Botmart, *Strong convergence of a new hybrid iterative scheme for nonexpansive mappings and application*, Journal of Function Spaces, **2022**, https://doi.org/10.1155/2022/4855173, 2022.

[13] M. A. Krasnosel'skii, *Two remarks on the method of successive approximations*, Uspekhi matematicheskikh nauk, **10**:1, 123–127, 1955.

[14] S. H. Khan, *A Picard-Mann hybrid iterative process*, Fixed Point Theory and Application, **2013**, 69, doi:10.1186/1687-1812-2013-69, 2013.

[15] W. R. Mann, *Mean value method in iteration*, Proc. Amer. Math. Soc., **4**, 506-510, 1953.

[16] C. Moore, *A double-sequence iteration process for fixed points of continuous pseudocontractions*, Computers & Mathematics with Applications,**43**:12, 1585–1589, 2002.

[17] M. A. Noor, *New approximation scheme for general variational inequalities*, Journal of Mathematical Analysis and Application, **251**, 217–229, doi:10.1006/jmaa.2000.7042, 2000.

[18] G. A. Okeke and M. Abbas, *A solution of delay differential equations via Picard–Krasnoselskii hybrid iterative process*,Arabian Journal of Mathematics,**6**, 21–29, 2017.

[19] G. A. Okeke, *Convergence analysis of the Picard–Ishikawa hybrid iterative process with applications*, Afrika Matematika, **30**, 817–835, 2019.

[20] G. A. Okeke and A. E. Ofem, *A novel iterative scheme for solving delay differential equations and nonlinear integral equations in Banach spaces*, Math. Meth. Appl. Sci., 1–24, https://doi.org/10.1002/mma.8095, 2022.

[21] G.A. Okeke and C.I. Ugwuogor, *Iterative construction of the fixed point of Suzuki's generalized nonexpansive mappings in Banach spaces,* Fixed Point Theory, 23, No. 2, 633-652, 2022.

[22] G.A. Okeke and C.I. Ugwuogor, *Convergence analysis of a new implicit iteration process for asymptotically nonexpansive mappings in the intermediate sense in Banach spaces,* The Journal of Analysis, 31, 1401-1423, 2023.

[23] È. Picard,*Memoire sur la theorie des equations aux derivees partielles et la methode des approximations successives*, Journal de Mathématiques Pures et Appliquées, **6**, 145–210, 1890.

[24] B. Rhoades and Z. Xue, *Comparison of the rate of convergence among Picard, Mann, Ishikawa, and Noor iterations applied to quasicontractive maps*, Fixed Point Theory and Applications, **2010**, 1–12, 2010.

[25] Ş. M. Şultuz and T. Grosan, *Data dependence for Ishikawa iteration when dealing with contractive-like operators*, Fixed Point Theory and Application, **2008**, 1–7, doi:10.1155/2008/242916, 2008.

[26] Şoltuz, Ştefan M and Otrocol, Diana *Classical results via Mann-Ishikawa iteration*, Revue d'analyse numérique et de théorie de l'approximation, **36**:2, 193–197, 2007.

[27] B. S. Thakur, D. Thakur and M. Postolache, *A new iterative scheme for numerical reckoning fixed points of Suzuki's generalized nonexpansive mappings*, Applied Mathematics and Computation, **275**, 147–155, http://dx.doi.org/10.1016/j.amc.2015.11.065, 2016.

[28] X. Weng, *Fixed point iteration for local strictly pseudo-contractive mapping*, Proceeding of the American Mathematical Society, **113**:3, 727–731, 1991.

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