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SOME FIXED POINT THEOREMS IN CONE RECTANGULAR METRIC SPACES

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ABSTRACT. We establish new fixed point theorems in cone rectangular metric spaces. The presented theorems generalize, extend and improve some existing results in the literature including the results of M. Jleli and B. Samet (2009), A. Azam and M. Arshad (2008), S. Moradi (2009), L.G. Huang and X. Zhang (2007), Sh. Rezapour and R. Hamlbarani (2008), I. Sahin and M. Telci (2009), and others. In all our results, we dispense with the the normality assumption which is a characteristic of most of the previous results.

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1. INTRODUCTION AND PRELIMINARIES

The concept of cone metric spaces was introduced by Huang and Zhang [7], where the set of real numbers is replaced by an ordered Banach space. They introduced the basic definitions and discuss some properties of convergence of sequences in cone metric spaces. They also obtained some fixed point theorems in normal cone metric spaces for mappings satisfying various contractive conditions. After that, cone metric spaces have been studied by many other authors (see [1, 2, 6, 8, 9, 10, 13, 14, 16] and others).

Following the idea of Branciari [5], Azam, Arshad and Beg [3] introduced the notion of cone rectangular metric spaces by replacing the triangle inequality with a rectangular inequality. After that, Jleli and Samet [11] extended the Kannan's fixed point theorem in such spaces.

Let E always be a real Banach space equipped with the norm $\|\cdot\|_E$ and P be a subset of E. We denote by 0_E the zero vector of E. P is called a cone if

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(i) P is closed, non-empty, $P \neq \{0_E\}$,
(ii) $a, b \in \mathbb{R}$, $a, b \geq 0$, $x, y \in P$ imply $ax + by \in P$,
(iii) $P \cap (-P) = \{0_E\}$.

For a given cone $P \subseteq E$, we can define a partial ordering $\leq_E$ on E with respect to P by:

$$x \leq_E y \Leftrightarrow y - x \in P \text{ for all } x, y \in E.$$

We shall write $x <_E y$ if $x \leq_E y$ and $x \neq y$, while $x \ll y$ will stand for $y - x \in \text{ int } P$, where $\text{ int } P$ denotes the interior of P.

The cone P is called normal if there exists $K > 0$ such that for all $x, y \in E$, we have:

$$0 \leq_E x \leq_E y \Rightarrow \|x\|_E \leq_E K\|y\|_E.$$

The least positive number K satisfying the above property is called the normal constant of P.

Rezapour and Hamlbarani [13] showed that there are no normal cones with normal constant $K < 1$. Moreover, they proved that for each $\nu > 1$ there are cones with normal constant $K > \nu$.

In the following, we always suppose that E is a real Banach space and P is a cone in E with $\text{ int } P \neq \emptyset$ and $\leq_E$ is the partial ordering with respect to P.

Definition 1.1. ([7]) Let X be a non-empty set. Suppose $\rho : X \times X \to E$ satisfies:

(1) $0_E \leq_E \rho(x, y)$ for all $x, y \in X$ and $\rho(x, y) = 0_E$ if and only if $x = y$,
(2) $\rho(x, y) = \rho(y, x)$ for all $x, y \in X$,
(3) $\rho(x, y) \leq_E \rho(x, z) + \rho(z, y)$ for all $x, y, z \in X$.

Then ρ is called a cone metric on X, and (X, ρ) is called a cone metric space.

Example 1.1. ([7]) Let $E = \mathbb{R}^2$, $P = \{(x, y) \in \mathbb{R}^2 \,|\, x \geq 0,\, y \geq 0\}$, $X = \mathbb{R}$ and $\rho : X \times X \to E$ defined by:

$$\rho(x, y) = (|x - y|, \tau|x - y|) \text{ for all } x, y \in X,$$

where $\tau \geq 0$ is a constant. Then (X, ρ) is a cone metric space.

Definition 1.2. ([3]) Let X be a non-empty set. Suppose $d : X \times X \to E$ satisfies:

(a) $0_E \leq_E d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0_E$ if and only if $x = y$,
(b) $d(x, y) = d(y, x)$ for all $x, y \in X$,

(c) $d(x,y) \leq_E d(x,w)+d(w,z)+d(z,y)$ for all $x,y \in X$ and for all distinct points $w,z \in X\backslash\{x,y\}$ [rectangular inequality].

Then d is called a cone rectangular metric on X, and (X,d) is called a cone rectangular metric space.

It is clear that any cone metric space is a cone rectangular metric space. The converse is not true in general (a counter-example is given in [11]).

Definition 1.3. ([3]) Let (X,d) be a cone rectangular metric space, $\{x_n\}$ be a sequence in X and $x \in X$. If for every $c \in E$ with $0_E \ll c$ there is $N \in \mathbb{N}$ such that for all $n > N$, $d(x_n,x) \ll c$, then $\{x_n\}$ is said to be convergent to x and x is a limit of $\{x_n\}$. We denote this by $x_n \to x$ as $n \to +\infty$ or $\lim_{n\to+\infty} x_n = x$.

Definition 1.4. ([3]) Let (X,d) be a cone rectangular metric space and $\{x_n\}$ be a sequence in X. If for all $c \in E$ with $0_E \ll c$ there is $N \in \mathbb{N}$ such that for all $n > N$, $d(x_n,x_{n+m}) \ll c$, then $\{x_n\}$ is called a Cauchy sequence in (X,d). If every Cauchy sequence is convergent in (X,d), then (X,d) is called a complete cone rectangular metric space.

Remark 1.1. The reader should give attention to the difference between cone metric spaces and cone rectangular metric spaces.
If (X,d) is a cone metric space and $\{x_n\}$ is a convergent sequence in (X,d), then the limit of $\{x_n\}$ is unique (see [7]-Lemma 2). However, when (X,d) is a cone rectangular metric space, it is not the case. A counter-example is given in [11] (see also [15]). If (X,d) is a cone metric space and $\{x_n\}$ is a convergent sequence in (X,d), then $\{x_n\}$ is a Cauchy sequence in (X,d) (see [7]-Lemma 3). However, when (X,d) is a cone rectangular metric space, this result is not true in general. A counter-example is given in [11] (see also 15]).

In this paper, we establish new fixed point theorems in rectangular cone metric spaces. Our obtained results generalize, extend and improve some existing results in the literature.

2. MAIN RESUTS

We need the following definition.

Definition 2.1. ([4]) Let (X,d) be a rectangular cone metric space. A mapping $S : X \to X$ is said to be sequentially convergent if we have, for every sequence $\{y_n\}$, if $\{Ty_n\}$ is convergent then $\{y_n\}$ is also convergent. S is said to be subsequentially convergent if we have, for every sequence $\{y_n\}$, if $\{Ty_n\}$ is convergent then $\{y_n\}$ has a convergent subsequence.

The proof of the following theorem essentially follows the classical case.

Theorem 2.1. Let (X, d) be a rectangular cone metric space. A mapping $S : X \to X$ is said to be continuous if and only if for every sequence $\{x_n\}$ in X, we have $x_n \to x$ implies $Sx_n \to Sx$.

Our first result is the following.

Theorem 2.2. Let (X, d) be a Hausdorff and complete cone rectangular metric space with cone P. Let $T, S : X \to X$ be mappings such that S is continuous, one-to-one and subsequentially convergent. Suppose that

$$d(STx, STy) \leq_E h[d(Sx, STx) + d(Sy, STy)] \tag{1}$$

for all $x, y \in X$, where $0 < h < 1/2$. Then, T has a unique fixed point. Moreover, if S is sequentially convergent, then for every $x_0 \in X$, the sequence $\{T^n x_0\}$ converges to this fixed point.

Proof. For any arbitrary point $x_0 \in X$, construct the sequence $\{x_n\}$ in X such that

$$x_{n+1} = Tx_n (\text{equivalently } x_n = T^n x_0) \text{ for all } n \in \mathbb{N}.$$

From (1), we have:

$$\begin{aligned} d(Sx_{n+1}, Sx_n) &= d(STx_n, STx_{n-1}) \\ &\leq_E h[d(Sx_n, STx_n) + d(Sx_{n-1}, STx_{n-1})] \\ &= h[d(Sx_n, Sx_{n+1}) + d(Sx_{n-1}, Sx_n)]. \end{aligned}$$

Then

$$d(Sx_{n+1}, Sx_n) \leq_E \left(\frac{h}{1-h}\right) d(Sx_{n-1}, Sx_n).$$

Continuing this process, we obtain:

$$d(Sx_{n+1}, Sx_n) \leq_E r^n d(Sx_0, Sx_1), \tag{2}$$

where

$$0 \leq r = \frac{h}{1-h} < 1.$$

We divide the proof into two cases.

• First case. Suppose that $Sx_m = Sx_n$ for some $m, n \in \mathbb{N}$, $m \neq n$. Let $m > n$, $p = m - n$ and $y = x_n = T^n x_0$. Since S is one-to-one, we have $x_m = x_n$, that is, $T^m x_0 = T^n x_0$, that is, $T^{m-n}(T^n x_0) = T^n x_0$, i.e., $T^p y = y$. By the same argument, we have:

$$d(Sy, STy) = d(ST^p y, ST^{p+1} y) \leq_E r^p d(Sy, STy).$$

Then

$$-(1-r^p)d(Sy, STy) \in P \quad \text{and} \quad (1-r^p)d(Sy, STy) \in P(\text{since } p > 1).$$

This implies by the definition of a cone that $d(Sy, STy) = 0_E$, that is, $Sy = STy$. Since S is one-to-one, we get $y = Ty$, and y is a fixed point of T.

• Second case. Suppose that $Sx_m \neq Sx_n$ for all $m, n \in \mathbb{N}$ with $m \neq n$. Using (1) and (2), we have:

$$\begin{array}{rcl} d(Sx_n, Sx_{n+2}) & = & d(STx_{n-1}, STx_{n+1}) \\ & \leq_E & h[d(Sx_{n-1}, STx_{n-1}) + d(Sx_{n+1}, STx_{n+1})] \\ & = & hd(Sx_{n-1}, Sx_n) + hd(Sx_{n+1}, Sx_{n+2}) \\ & \leq_E & hr^{n-1}(1 + r^2)d(Sx_0, Sx_1) \\ & \leq_E & hr^{n-1}(1 + r)d(Sx_0, Sx_1) \\ & \leq_E & r^n d(Sx_0, Sx_1). \end{array}$$

Thus,

$$d(Sx_n, Sx_{n+2}) \leq_E r^n d(Sx_0, Sx_1). \tag{3}$$

Now, if $m > 2$ is odd then writing $m = 2\ell + 1$, $\ell \geq 1$, using the rectangular inequality and (2), we can easily show that

$$\begin{array}{rcl} d(Sx_n, Sx_{n+m}) & \leq_E & d(Sx_n, Sx_{n+1}) + d(Sx_{n+1}, Sx_{n+2}) + \cdots \\ & & +d(Sx_{n+2\ell}, Sx_{n+2\ell+1}) \\ & \leq_E & (r^n + r^{n+1} + \cdots + r^{n+2\ell})d(Sx_0, Sx_1) \\ & \leq_E & \dfrac{r^n}{1-r}\, d(Sx_0, Sx_1). \end{array}$$

If $m > 2$ is even then writing $m = 2\ell$, $\ell \geq 2$, using the rectangular inequality, (2) and (3), we get:

$$\begin{array}{rcl} d(Sx_n, Sx_{n+m}) & \leq_E & d(Sx_n, Sx_{n+2}) + d(Sx_{n+2}, Sx_{n+3}) + \cdots \\ & & +d(Sx_{n+2\ell-1}, Sx_{n+2\ell}x_0) \\ & \leq_E & (r^n + r^{n+2} + r^{n+3} + \cdots + r^{n+2\ell-1})d(Sx_0, Sx_1) \\ & \leq_E & \dfrac{r^n}{1-r}\, d(Sx_0, Sx_1). \end{array}$$

Thus combining all the cases, we obtain:

$$d(Sx_n, Sx_{n+m}) \leq_E \frac{r^n}{1-r} d(Sx_0, Sx_1) \text{ for all } m, n \in \mathbb{N}. \qquad (4)$$

Now, let c ba an arbitrary point in E with $0_E \ll c$. Since, $0 < r < 1$, there exists $N \in \mathbb{N}$ such that

$$\frac{r^n}{1-r} d(Sx_0, Sx_1) \ll c \text{ for all } n > N. \qquad (5)$$

Combining (4) and (5), we obtain:

$$d(Sx_n, Sx_p) \ll c \text{ for all } p > n > N.$$

Then we proved that $\{Sx_n\}$ is a Cauchy sequence in (X, d). Since (X, d) is a complete rectangular cone metric space, there is $\overline{x} \in X$ such that

$$Sx_n \to \overline{x} \text{ as } n \to +\infty. \qquad (6)$$

Since S is subsequentially convergent, $\{x_n\}$ has a convergent subsequence. Then there exists $x^* \in X$ such that $x_{n(p)} \to x^*$ as $p \to +\infty$. Using the continuity of S, we have:

$$Sx_{n(p)} \to Sx^* \quad \text{as } p \to +\infty. \qquad (7)$$

Combining (6) and (7), and using the fact that (X, d) is Hausdorff, we have:

$$Sx^* = \overline{x}. \qquad (8)$$

Now, let c be an arbitrary point of E such that $0_E \ll c$. From (2), (6) and (8), there exists $N \in \mathbb{N}$ such that

$$d(Sx^*, Sx_n) + (1+h)d(Sx_n, Sx_{n+1}) \ll (1-h)c \text{ for all } n \geq N. \quad (9)$$

Without any loss of generality, we can assume that $Sx_r \neq Sx^*, STx^*$ for all $r \in \mathbb{N}$. Using (1) and the rectangular inequality, we have:

$$\begin{aligned} d(Sx^*, STx^*) &\leq_E d(Sx^*, Sx_n) + d(Sx_n, Sx_{n+1}) + d(STx_n, STx^*) \\ &\leq_E d(Sx^*, Sx_n) + d(Sx_n, Sx_{n+1}) + hd(Sx_n, Sx_{n+1}) \\ &\quad +hd(Sx^*, STx^*). \end{aligned}$$

Then, from (9), for all $n \geq N$, we have:

$$d(Sx^*, STx^*) \leq_E \frac{1}{1-h}[d(Sx^*, Sx_n) + (1+h)d(Sx_n, Sx_{n+1})] \ll c.$$

Therefore,

$$d(Sx^*, STx^*) \leq_E c \text{ for all } c \gg 0_E.$$

Hence, for a fixed $c \gg 0_E$, we have:

$$\varepsilon c - d(Sx^*, STx^*) \in P \text{ for all } \varepsilon > 0.$$

Letting $\varepsilon \to 0^+$ and using that P is closed, we obtain:

$$-d(Sx^*, STx^*) \in P \quad \text{and} \quad d(Sx^*, STx^*) \in P.$$

Thus, $d(Sx^*, STx^*) = 0_E$, that is, $Sx^* = STx^*$. Since S is one-to-one, we get $x^* = Tx^*$, and x^* is a fixed point of T.

Now, suppose that y^* is another fixed point of T, that is, $y^* = Ty^*$. From (1), we have:

$$d(Sx^*, Sy^*) = d(STx^*, STy^*) \leq_E hd(Sx^*, Sx^*) + hd(Sy^*, Sy^*) = 0_E.$$

Then, $Sx^* = Sy^*$ and $x^* = y^*$ (since S is one-to-one). Then we proved the uniqueness of the fixed point of T.

Now, if S is sequentially convergent, by replacing $\{n(p)\}$ with $\{n\}$, we conclude that $x_n \to x^*$ and this shows that $\{x_n\}$ converges to the fixed point of T. □

Remark 2.1. Taking $S : X \to X$ the identity mapping ($Sx = x$ for all $x \in X$) in Theorem 2. 2, we obtain the result given by Jleli and Samet in [11].

The following example shows that Theorem 2.2 is indeed a proper extension of Jleli and Samet Theorem [11]. This example is inspired by [12].

Example 2.1. Let $X = \{1, 2, 3, 4\}$. Denote by $\mathcal{M}_k(\mathbb{R})$ the set of $k \times k$ real matrices, $k \in \mathbb{N}$. Let $E = \mathcal{M}_k(\mathbb{R})$ and $P = \{(a_{ij})_{1 \leq i,j \leq k} \in E \,|\, a_{ij} \geq 0 \text{ for all } i, j\}$. We denote by I_k the identity matrix and 0_E the zero matrix. Define $d : X \times X \to \mathcal{M}_k(\mathbb{R})$ by:

$$\begin{aligned} d(1,2) &= d(2,1) = 3I_k, \\ d(2,3) &= d(3,2) = d(1,3) = d(3,1) = I_k, \\ d(1,4) &= d(4,1) = d(2,4) = d(4,2) = d(3,4) = d(4,3) = 4I_k, \\ d(1,1) &= d(2,2) = d(3,3) = d(4,4) = 0_E. \end{aligned}$$

Obviously (X, d) is a rectangular cone metric space and is not a cone metric space since

$$d(1,2) = 3I_k >_E d(1,3) + d(3,2) = I_k + I_k = 2I_k.$$

Define $T : X \to X$ by:

$$Tx = \begin{cases} 2 & \text{if} \quad x \neq 1, \\ 4 & \text{if} \quad x = 1. \end{cases}$$

Now, define $S : X \to X$ by:

$$Sx = \begin{cases} 2 & \text{if} \quad x = 4, \\ 3 & \text{if} \quad x = 2, \\ 4 & \text{if} \quad x = 1, \\ 1 & \text{if} \quad x = 3. \end{cases}$$

Now, let us check that all the hypotheses of Theorem 2. 2 are satisfied.

• (X, d) is Hausdorff.
Let $\{x_n\}$ be a sequence in X such that $x_n \to x$ and $x_n \to y$ as $n \to +\infty$, where $x, y \in X$. We have:

$$d(x_n, x), d(x_n, y) \in \{0_E, I_k, 3I_k, 4I_k\}.$$

Let c be a $k \times k$ matrix such that the entries are non-zero and the diagonal entries less than 1. Since $c \gg 0_E$, there exists $N \in \mathbb{N}$ such that

$$d(x_n, x) \ll c \text{ for all } n \geq N \quad \text{and} \quad d(x_n, y) \ll c \text{ for all } n \geq N.$$

This implies that

$$d(x_n, x) = d(x_n, y) = 0_E \text{ for all } n \geq N.$$

Then $x = y$ and (X, d) is a Hausdorff cone rectangular metric space.

• (X, d) is complete.
Let $\{x_n\}$ be a Cauchy sequence in (X, d). Let c be as defined above. Since $c \gg 0_E$, there exists $N \in \mathbb{N}$ such that

$$d(x_n, x_m) \ll c \text{ for all } m > n > N.$$

Since $d(x_n, x_m) \in \{0_E, I_k, 3I_k, 4I_k\}$, we have:

$$d(x_n, x_m) = 0_E \text{ for all } m > n > N.$$

This implies that

$$x_n = x_{N+1} \text{ for all } n \geq N + 1.$$

Therefore

$$x_n \to x_{N+1} \text{ as } n \to +\infty.$$

Then (X, d) is complete.

• $S : X \to X$ is continuous and one-to-one.
Let $\{x_n\}$ be a sequence in X such that $x_n \to x \in X$ as $n \to +\infty$. This implies that there exists $N \in \mathbb{N}$ such that $x_n = x$ for all $n \geq N$. Then $Sx_n \to Sx$ as $n \to +\infty$, and S is continuous. It is

immediate to show that S is one-to-one.

• S is sequentially convergent.
Let $\{x_n\}$ be a sequence in X such that $Sx_n \to x \in X$ as $n \to +\infty$. Then there exists $N \in \mathbb{N}$ such that $Sx_n = x$ for all $n \geq N$. Since S is one-to-one, we have $x_n = x_N$ for all $n \geq N$. Then $\{x_n\}$ is convergent, and S is sequentially convergent.

• The required contractive condition holds.
For all $x \in X$, we have:

$$STx = \begin{cases} 3 & \text{if} \quad x \neq 1, \\ 2 & \text{if} \quad x = 1. \end{cases}$$

We can show that

$$d(STx, STy) \leq_E \frac{1}{3}[d(Sx, STx) + d(Sy, STy)]$$

for all $x, y \in X$.

Now, all the hypotheses of Theorem 2. 2 are satisfied. Then T has a unique fixed point, that is, $x^* = 2$. Note that Jleli and Samet theorem [11] can not be used in this case, since we have:

$$d(T1, T2) = d(1, T1) + d(2, T2).$$

Remark 2.2. Theorem 2. 2 extends given results in [2, 7, 12, 13].

We use the following notation. For $a, b, c \in E$, we will denote the proposition $(a \leq_E b) \vee (a \leq_E c)$ by $a \leq_E [b \bigvee c]$. Moreover, for $r \in \mathbb{R}$, $a \leq_E r\,[b \bigvee c]$ will denote $(a \leq_E rb) \vee (a \leq_E rc)$.

Corollary 2.1. Let (X, d) be a a Hausdorff and complete cone rectangular metric space with cone P. Let $T, S : X \to X$ be mappings such that S is continuous, one-to-one and subsequentially convergent. Suppose that

$$d(STx, STy) \leq_E h \left[d(Sx, STx) \bigvee d(Sy, STy)\right] \tag{10}$$

for all $x, y \in X$, where $0 < h < 1/2$. Then, T has a unique fixed point. Moreover, if S is sequentially convergent, then for every $x_0 \in X$, the sequence $\{T^n x_0\}$ converges to this fixed point.

Proof. It is easy to show that (10) implies (1). Then, the result is an immediate consequence of Theorem2. 2. □

Now, we address the following question: Does the result of Corollary 2. 1 hold if $h \geq \frac{1}{2}$? The next theorem gives a positive answer in the case $\frac{1}{2} \leq h < 1$.

Theorem 2.3. Let (X,d) be a a Hausdorff and complete cone rectangular metric space with cone P. Let $T,S : X \to X$ be mappings such that S is continuous, one-to-one and subsequentially convergent. Suppose that

$$d(STx,STy) \leq_E h\left[d(STx,Sx)\bigvee d(STy,Sy)\right] \tag{11}$$

for all $x,y \in X$, where $0 < h < 1$. Then T has a unique fixed point.

Proof. For any arbitrary point $x_0 \in X$, construct the sequence $\{x_n\}$ in X such that

$$x_{n+1} = Tx_n \text{ for all } n \in \mathbb{N}.$$

Without restriction to the generality, we can suppose that $x_n \neq x_{n+1}$ for all n. From (11), we have:

$$\begin{aligned} d(Sx_{n+1},Sx_n) &= d(STx_n,STx_{n-1}) \\ &\leq_E h\left[d(Sx_{n+1},Sx_n)\bigvee d(Sx_n,Sx_{n-1})\right]. \end{aligned}$$

If $d(Sx_{n+1},Sx_n) \leq_E hd(Sx_{n+1},Sx_n)$, we have $-(1-h)d(Sx_{n+1},Sx_n) \in P$. Since $h < 1$, also we have $(1-h)d(Sx_{n+1},Sx_n) \in P$. By the definition of a cone, we get $Sx_n = Sx_{n+1}$, which implies since S is one-to-one that $x_n = x_{n+1}$, that is impossible by assumption. Then

$$d(Sx_{n+1},Sx_n) \leq_E hd(Sx_n,Sx_{n-1}).$$

Continuing this process, we obtain:

$$d(Sx_{n+1},Sx_n) \leq_E h^n d(Sx_0,Sx_1) \text{ for all } n. \tag{12}$$

We divide the proof into two cases.

• First case. Suppose that $x_m = x_n$ for some $m,n \in \mathbb{N}$, $m \neq n$. Let $m > n$, then $T^{m-n}(T^n x_0) = T^n x_0$, that is, $T^p y = y$, where $p = m-n$ and $y = T^n x_0$. From (12), we have:

$$d(Sy,STy) = d(ST^p y, ST^{p+1}y) \leq_E h^p d(Sy,STy).$$

Since, $p > 1$ and $h < 1$, we obtain $d(Sy,STy) = 0_E$, that is, $Sy = STy$, which implies since S is one-to-one that y is a fixed point of T.

• Second case. Suppose that $x_m \neq x_n$ for all $m,n \in \mathbb{N}$ with $m \neq n$. Using (11), we have:

$$\begin{aligned} d(Sx_n,Sx_{n+2}) &= d(STx_{n-1},STx_{n+1}) \\ &\leq_E h\left[d(Sx_{n-1},Sx_n)\bigvee d(Sx_{n+2},Sx_{n+1})\right]. \end{aligned}$$

If $d(Sx_n, Sx_{n+2}) \leq_E hd(Sx_{n-1}, Sx_n)$, from (12), we have:

$$\begin{aligned} d(Sx_n, Sx_{n+2}) &\leq_E hd(Sx_{n-1}, Sx_n) \\ &\leq_E h^n d(Sx_0, Sx_1). \end{aligned}$$

If $d(Sx_n, Sx_{n+2}) \leq_E hd(Sx_{n+2}, Sx_{n+1})$, we have:

$$\begin{aligned} d(Sx_n, Sx_{n+2}) &\leq_E hd(Sx_{n+2}, Sx_{n+1}) \\ &\leq_E h^{n+2} d(Sx_0, Sx_1) \\ &\leq_E h^n d(Sx_0, Sx_1). \end{aligned}$$

Then, in all cases, we have:

$$d(Sx_n, Sx_{n+2}) \leq_E h^n d(Sx_0, Sx_1). \tag{13}$$

Now, if $m > 2$ is odd then writing $m = 2\ell + 1$, $\ell \geq 1$, using the rectangular inequality and (12), we can easily show that

$$\begin{aligned} d(Sx_n, Sx_{n+m}) &\leq_E d(Sx_n, Sx_{n+1}) + d(Sx_{n+1}, Sx_{n+2}) + \cdots \\ &\quad + d(Sx_{n+2\ell}, Sx_{n+2\ell+1}) \\ &\leq_E (h^n + h^{n+1} + \cdots + h^{n+2\ell}) d(Sx_0, Sx_1) \\ &\leq_E \frac{h^n}{1-h} d(Sx_0, Sx_1). \end{aligned}$$

If $m > 2$ is even then writing $m = 2\ell$, $\ell \geq 2$, using the rectangular inequality, (12) and (13), we get:

$$\begin{aligned} d(Sx_n, Sx_{n+m}) &\leq_E d(Sx_n, Sx_{n+2}) + d(Sx_{n+2}, Sx_{n+3}) + \cdots \\ &\quad + d(Sx_{n+2\ell-1}, Sx_{n+2\ell}) \\ &\leq_E (h^n + h^{n+2} + h^{n+3} + \cdots + h^{n+2\ell-1}) d(Sx_0, Sx_1) \\ &\leq_E \frac{h^n}{1-h} d(Sx_0, Sx_1). \end{aligned}$$

Thus combining all the cases, we obtain:

$$d(Sx_n, Sx_{n+m}) \leq_E \frac{h^n}{1-h} d(Sx_0, Sx_1) \text{ for all } m, n \in \mathbb{N}. \tag{14}$$

Now, let c ba an arbitrary point in E with $0_E \ll c$. Since $0 < h < 1$, there exists $N \in \mathbb{N}$ such that

$$\frac{h^n}{1-h} d(Sx_0, Sx_1) \ll c \text{ for all } n > N. \tag{15}$$

Combining (14) and (15), we obtain:

$$d(Sx_n, Sx_p) \ll c \text{ for all } p > n > N.$$

Then we proved that $\{Sx_n\}$ is a Cauchy sequence in (X,d). Since (X,d) is a complete rectangular cone metric space, there is $\overline{x}\in X$ such that

$$Sx_n \to \overline{x} \text{ as } n\to+\infty. \tag{16}$$

Since S is subsequentially convergent, $\{x_n\}$ has a convergent subsequence. Then there exists $x^*\in X$ such that $x_{n(p)}\to x^*$ as $p\to+\infty$. Using the continuity of S, we have:

$$Sx_{n(p)}\to Sx^* \quad \text{as } p\to+\infty. \tag{17}$$

Combining (16) and (17) and using that (X,d) is Hausdorff, we have:

$$Sx^*=\overline{x}. \tag{18}$$

Let c be an arbitrary point in E with $0_E\ll c$. From (12), (16) and (18), there is $N\in\mathbb{N}$ such that

$$d(Sx^*,Sx_n)+d(Sx_n,Sx_{n+1})+hd(Sx_n,Sx_{n+1})\ll(1-h)c \tag{19}$$

for all $n\geq N$. Without any loss of generality, we can assume that $Sx_r\neq Sx^*, STx^*$ for all $r\in\mathbb{N}$. Using (11) and the rectangular inequality, we obtain:

$$\begin{aligned} d(Sx^*,STx^*) &\leq_E d(Sx^*,Sx_n)+d(Sx_n,Sx_{n+1})+d(STx_n,STx^*)\\ &\leq_E d(Sx^*,Sx_n)+d(Sx_n,Sx_{n+1})\\ &\quad +h\left[d(Sx_n,Sx_{n+1})\bigvee d(STx^*,Sx^*)\right]\\ &\leq_E d(Sx^*,Sx_n)+d(Sx_n,Sx_{n+1})\\ &\quad +hd(Sx_n,Sx_{n+1})+hd(STx^*,Sx^*). \end{aligned}$$

Therefore, by (19), for all $n\geq N$, we have:

$$d(Sx^*,STx^*)\leq_E \frac{1}{1-h}[d(Sx^*,Sx_n)+d(Sx_n,Sx_{n+1})+hd(Sx_n,Sx_{n+1})]\ll c.$$

Then we proved that

$$c-d(Sx^*,STx^*)\in P \text{ for all } c\gg 0_E.$$

Then, for a fixed $c\gg 0_E$, we have:

$$\varepsilon c-d(Sx^*,STx^*)\in P \text{ for all } \varepsilon>0.$$

Letting $\varepsilon\to 0^+$, since P is closed, we obtain:

$$-d(Sx^*,STx^*)\in P \quad\text{and}\quad d(Sx^*,STx^*)\in P.$$

Thus $d(Sx^*,STx^*)=0_E$, that is, $Sx^*=STx^*$. Since S is one-to-one, we have $x^*=Tx^*$, and x^* is a fixed point of T. Now, suppose that y^* is also a fixed point of T, that is, $y^* = Ty^*$. Using (11), we have:

$$d(Sx^*, Sy^*) = d(STx^*, STy^*) \leq_E h \left[d(Sx^*, STx^*) \bigvee d(Sy^*, STy^*) \right] = 0_E.$$

Then $Sx^* = Sy^*$ and since S is one-to-one, we have $x^* = y^*$. Hence, the uniqueness of the fixed point is proved.

Now, if S is sequentially convergent, by replacing $\{n(p)\}$ with $\{n\}$, we conclude that $x_n \to x^*$ and this shows that $\{x_n\}$ converges to the fixed point of T. □

Taking $S : X \to X$ the identity mapping ($Sx = x$ for all $x \in X$) in Theorem 2.3, we obtain the following result.

Corollary 2.2. Let (X, d) be a Hausdorff and complete cone rectangular metric space with cone P. Let $T : X \to X$ be a mapping satisfying:

$$d(Tx, Ty) \leq_E h \left[d(Tx, x) \bigvee d(Ty, y) \right]$$

for all $x, y \in X$, where $0 < h < 1$. Then T has a unique fixed point.

Remark 2.3. Corollary 2.2 is an extension of a result of Sahin and Telci (Theorem 4.3, [14]).

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REFERENCES

[1] M. Abbas, B. E. Rhoades and T. Nazir, *Common fixed points for four maps in cone metric spaces*, Applied Mathematics and Computation. **216** 80-86, 2010.

[2] A. Azam and M. Arshad, *Kannan fixed point theorem on generalized metric spaces*, J. Nonlinear Sci. Appl. **1** (1) 45-48, 2008.

[3] A. Azam, M. Arshad and I. Beg, *Banach contraction principle on cone rectangular metric spaces*, Appl. Anal. Discrete Math. **3** 236-241, 2009.

[4] A. Beiranvand, S. Moradi, M. Omid and H. Pazandeh, *Two fixed point theorems for special mappings*, arXiv:0903.1504v1 [math.FA].

[5] A. Branciari, *A fixed point theorem of Banach-Caccioppoli type on a class of generalized metric spaces*, Publ. Math. Debrecen. **57** 1-2 31-37, 2000.

[6] Wei-Shih Du, *A note on cone metric fixed point theory and its equivalence*, Nonlinear Analysis. **72** 2259-2261, 2010.

[7] L. G. Huang and X. Zhang, *Cone metric spaces and fixed point theorems of contractive mappings*, J. Math. Anal. Appl. **332** 1468-1476, 2007.

[8] D. Ilić and V. Rakočević, *Common fixed points for maps on cone metric space*, J. Math. Anal. Appl. **341** 876-882, 2008.

[9] S. Janković, Z. Golubović and S. Radenović, *Compatible and weakly compatible mappings in cone metric spaces*, Mathematical and Computer Modelling, doi:10.1016/j.mcm.2010.06.043, 2010.

[10] S. Janković, Z. Kadelburg, S. Radenović and B. E. Rhoades, *Assad-Kirk-Type Fixed Point Theorems for a Pair of Nonself Mappings on Cone Metric Spaces*, Fixed Point Theory and Applications, Volume **2009**, Article ID 761086, 2009.

[11] M. Jleli and B. Samet, *The Kannan's fixed point theorem in a cone rectangular metric space*, J. Nonlinear Sci. Appl. **2** (3) 161-167, 2009.

[12] S. Moradi, *Kannan fixed point theorem on complete metric spaces and on generalized metric spaces depended on another function*, arXiv:0903.1577v1 [math.FA].

[13] Sh. Rezapour and R. Hamlbarani, *Some notes on paper cone metric spaces and fixed point theorems of contractive mappings*, J. Math. Anal. Appl. **345** 719-724, 2008.

[14] I. Sahin and M. Telci, *Fixed points of contractive mappings on complete cone metric spaces*, Hacettepe Journal of Mathematics and Statistics. **38** (1) 59-67, 2009.

[15] B. Samet, *Discussion on: A fixed point theorem of Banach-Caccioppoli type on a class of generalized metric spaces, by A. Branciari*, Publ. Math. Debrecen. **76**/4 493-494, 2010.

[16] P. Vetro, *Common fixed points in cone metric spaces*, Rend. Circ. Mat. Palermo. (2) **56** (3) 464-468, 2007.

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