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SOME EXAMPLES OF FINITE OSBORN LOOPS

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ABSTRACT. In this work we give a number of constructions of finite Osborn loops of order $4n$, with two generators. All the loops are found to satisfy both Langrange's theorem and Sylow's first theorem. They are found to be non-universal Osborn loops except when $k = 1$ and $n \leq 3$. Moreover, all the examples are found not to be flexible and do not have the LAP or RAP or LIP or RIP or AAIP, consequently not Moufang. The first three cases are particular examples for demonstration purpose. Therefore, finite Osborn loops of order 16, 24, 36, 48 and 72 has been constructed.

Keywords and phraces: Binary operations, Osborn loops, finite examples, non-universal
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In memory of a great Mathematician: Professor Haroon Oladipo Tejumola

1. INTRODUCTION

A loop $I(\cdot)$ is called an Osborn loop if it obeys the identity:

$$(x^{\lambda}\backslash y) \cdot zx = x(yz \cdot x) \tag{1}$$

for all $x, y, z \in I$. The term Osborn loops first appeared in a work of Huthnance Jr [3] in 1968, on generalized Moufang loops. However, the definition (1) above is according to Basarab [1] in 1979. For detail review of Osborn loops-see Isere *et al* submitted in another Journal [4], and other authors as [8],[12],[14]. Moreover, the most popularly known varieties of Osborn loops are CC-loops, Moufang loops, VD-loops and universal weak inverse property loops. All these four varieties of Osborn loops are universal [6]. This is what makes non-universal Osborn loops interesting to researchers like Kinyon, Phillips and others [5], [11]. Therefore, it would be a

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worthwhile effort to be able to construct a finite Osborn loop that is non-universal.

Following the pattern of example of finite Bol loops constructed by Solarin and Sharma [13], and CC-loops by Chiboka and Solarin [2], we develop new methods of constructing finite Osborn loops. We established that all the constructions are non-universal Osborn loops using the identity for universal Osborn loops derived by Jaiyeola and Adeniran in [5] as used in [6]. Strikingly, constructions 4 is a 'k' number of constructions where k is any integer. All of them are non-universal except when $k = 1$, and $n \leq 3$ where it gives a group, and atmost a Moufang loop respectively. In section 3, we prove that all the construtions to satisfy the Osborn identity, and for conservation of space, we shall prove one of the Osborn identity. In section 4, some properties of the constructed examples are presented.

2. PRELIMINARY

Definition 1. *A loop is a set I with binary operation (denoted here simply by juxtaposition) such that*

- *for each a in I, the left multiplication map $L_a : I \to I, x \to ax$ is bijective,*
- *for each a in I, the right multiplication map $R_a : I \to I, x \to xa$ is bijective; and*
- *I has a two-sided identity* 1.

The order of I is its cardinality $|I|$.

Definition 2. *A loop $(G, \cdot)$ is called, for all $x, y, z \in G$:* [10], [14]

(1) *a left inverse property loop (LIPL) if it has the left inverse property i.e. if there exists a bijection $J_\lambda : x \to x^\lambda$ on G such that $x^\lambda \cdot xy = y$*
(2) *a RIPL if it has the right inverse property(RIP)i.e. if there exists a bijection: $J_\rho : x \to x^\rho$ on G such that $yx \cdot x^\rho = y$*
(3) *an automorphic inverse property loop (AIPL) if and only if it obeys the identity: $(xy)^\lambda = x^\lambda y^\lambda$ or $(xy)^\rho = x^\rho y^\rho$*
(4) *a left alternative property loop (LAPL)if it obeys the left alternative property (LAP): $xx \cdot y = x \cdot xy$*
(5) *a right alternative property loop (RAPL) if it obeys the right alternative property (RAP): $y \cdot xx = yx \cdot x$.*
(6) *a conjugacy closed loop(CC-loop) if and only if it obeys $x \cdot yz = (xy)/(x \cdot xz)$ and $zy \cdot x = zx \cdot x\backslash(yx)$.*

(7) *a weak inverse property loop (WIPL) if it obeys the identity:* $x(yx)^\rho = y^\rho$ *or* $(xy)^\lambda x = y^\lambda$
(8) *a cross inverse property loop (CIPL) if and only if it obeys the identity:* $xy \cdot x^\rho = y$ *or* $x \cdot yx^\rho = y$ *OR* $x^\lambda \cdot (yx) = y$ *or* $x^\lambda y \cdot x = y$.
(9) *a flexible loop if and only if it obeys* $xy \cdot x = x \cdot yx$.
(10) *a power associative loop if and only if each element of* G *generates an associative subloop.*
(11) *a diassociative loop if and only if each pair generates an associative subloop.*
(12) *an anti-automorphic inverse property loop (AAIPL) if and only if it obeys the identity:* $(xy)^\rho = y^\rho x^\rho$ *or* $(xy)^\lambda = y^\lambda x^\lambda$.
(13) *a semi-automorphic inverse property loop (SAIPL) if and only if it obeys the identity:* $(xy{\cdot}x)^\rho = x^\rho y^\rho{\cdot}x^\rho$ *or* $(xy{\cdot}x)^\lambda = x^\lambda y^\lambda \cdot x^\lambda$.
(14) *the left nuclus of* G *is denoted by:*

$$N_\lambda(G,\cdot) = \{a \in G : ax \cdot y = a \cdot xy \ \forall \ x, y \in G\}.$$

(15) *the right nuclus of* G *is denoted:*

$$N_\rho(G,\cdot) = \{a \in G : y \cdot xa = yx \cdot a \ \forall \ x, y \in G\}.$$

(16) *the middle nuclus of* G *is denoted by:*

$$N_\mu(G,\cdot) = \{a \in G : ya \cdot x = y \cdot ax \ \forall \ x, y \in G\}.$$

(17) *the nuclus of* G *is denoted by:*

$$N(G,\cdot) = N_\lambda(G,\cdot) \cap N_\rho(G,\cdot) \cap N_\mu(G,\cdot).$$

(18) *the centrum of* G *denoted by:*

$$C(G,\cdot) = \{a \in G : ax = xa \ \forall \ x \in G\}.$$

(19) *the center of* G *is denoted by:*

$$Z(G,\cdot) = N(G,\cdot) \cap C(G,\cdot).$$

Theorem 1. *(Kinyon [7]) The smallest order for which proper (non-Moufang and non-CC) Osborn loops with non-trivial nucleus exists is* 16. *There are two of such loops.*

- *Each of the two is a G-loop.*
- *Each contains as a subgroup, the dihedral group* (D_4) *of order* 8.
- *For each loop, the center coincides with the nucleus and has order* 2. *The quotient by the center is a non-associative CC-loop of order* 8.
- *The second center is* $Z_2 \times Z$, *and the quotient is* Z_4.

- *One loop satisfies $L_x^4 = R_x^4 = I$, the other does not.*

AIP Osborn loops include :

- commutative Moufang loops and
- AIP CC-loops.

Lemma 1. *(Jaiyeola and Adeniran [5]) An Osborn loop that is flexible or which has the LAP or RAP or LIP or RIP or AAIP is a Moufang loop. But an Osborn loop that is commutative or which has the CIP is a Commutative Moufang loop.*

Lemma 2. *(Jaiyeola and Adeniran [5]) Let $(Q, \cdot, \backslash, /)$ be a left universal Osborn loop. The following identities are satisfied:*

$$v \cdot vv = v^{\lambda} \backslash v \cdot v \qquad and \qquad vv \cdot vv = v^{\lambda} \backslash (v^{\lambda^2} v) \cdot v$$

Theorem 2. *(Huthnance [3] and Basarab) Let G be an Osborn loop. $N_\rho(G) = N_\lambda(G) = N_\mu(G) = N(G) \trianglelefteq G$*

According to Huthnance [3], for a WIPL, the four nuclei N_ρ, N_λ, N_μ and N coincide, i.e.$N_\rho = N_\lambda = N_\mu = N$. The same statement is true for a CIPL and an IPL since they are WIPLs. This fact was observered also by Osborn-see [7],[6].

Example 1. *(Huthnance [3]) Let $H = Z \times Z \times Z$. Define a binary operation $\star$ on H by :*

$$[2i, k, m] \star [2j, p, q] = [2i + 2j, k + p - ij(2j - 1), q + m - ij(2j - 1)]$$

$$[2i + 1, k, m] \star [2j, p, q] = [2i + 2j + 1, k + p - ij(2j - 1) - j^2 + j \\ , q + m - ij(2j - 1) - j^2]$$

$$[2i, k, m] \star [2j+1, p, q] = [2i+2j+1, m+p-ij(2j+1), q+k-ij(2j+1)]$$

$$[2i + 1, k, m] \star [2j + 1, p, q] = [2i + 2j + 2, m + p - ij(2j + 1) - j^2 + j \\ , q + k - ij(2j + 1) - j^2]$$

$\forall\ i, j, k, m, p, q \in Z$. *$(H, \star)$ is an Osborn loop.*

Jaiyeola & Adeniran in 2009 [6] used this example to show that all Osborn loops are not universal. They began by saying: assuming that $(H, \star)$ is a universal Osborn loop, then it should obey the identity $v \cdot vv = v^{\lambda} \backslash v \cdot v$ [4]. Let $v = [2i + 1, k, m]$ then by direct computation, we have $v \cdot vv = [6i + 3, m + 2k - 10i^3 - 12i^2 - 2i, 2m + k - 10i^3 - 12i^2 - i - 1]$ and $v^{\lambda} \backslash v \cdot v = [6i + 3, m + 2k - 14i^3 - 18i^2 - 7i - 1, 2m + k - 14i^3 - 16i - 6i - 1]$. So, $v \cdot vv \neq v^{\lambda} \backslash v \cdot v$. Thus, $(H, \star)$ is not a universal Osborn loop-see [4].

MAIN RESULTS

Example 2. *Construction* 1
Let $I(\cdot) = C_{2n} \times C_2$ *that is* $I = \{(x^\alpha, p^\beta), 0 \leq \alpha \leq 2n-1, 0 \leq \beta \leq 1\}$ *and the binary operation is defined as follows:*

$$(x^a, e) \cdot (x^b, p^\beta) = (x^{a+b}, p^\beta)$$

$$(x^a, p^\alpha) \cdot (x^b, e) = (x^{a+b}, p^\alpha)$$

$$(x^a, p^\alpha) \cdot (x^b, p^\beta) = (x^{a+b}, p^{\alpha+\beta}) \ if \ b \equiv 0(mod\ 2)$$

$$= (x^{a+3b}, p^{\alpha+\beta}) \ if \ b \equiv 1(mod\ 2)$$

$$(x^a, p^\alpha) \cdot (x^b, p^\beta) = (x^{a+3b}, p^{\alpha+3\beta}) \ if \ a \equiv 1(mod\ 2), b \equiv 1(mod\ 2)$$

$$(x^{b+c}, p^\delta) \cdot (x^a, p^\alpha) = (x^{a+b+c}, p^{\alpha+\delta}) \ if \ b \equiv 0(mod\ 2)$$

$$(x^{b+c}, p^\delta) \cdot (x^a, p^\alpha) = (x^{a+3b+c}, p^{\alpha+\delta}) \ if \ b \equiv 1(mod\ 2)$$

$$(x^{b+c}, p^{\beta+\gamma}) \cdot (x^a, p^\alpha) = (x^{3a+3b+c}, p^{\alpha+3\beta+\gamma}) \ if \ a \equiv 1(mod\ 2)$$

$$, b \equiv 1(mod\ 2)$$

Then $I(\cdot)$ *is an Osborn loop of order* $4n$, *where* $n = 2, 3, 4, 6, 9, 12$ *and* 18

Proof
We first show that $I(\cdot)$ satisfies Osborn identity (1):

$$(x^\lambda \backslash y) \cdot zx = x(yz \cdot x) \ \forall \ x, y, z \in I$$

(a): Let $x = (x^a, e); y = (x^b, e); z = (x^c, e)$, then by direct computation we have

$$(x^\lambda \backslash y) \cdot zx = (x^{2a+b+c}, e)$$

$$x(yz \cdot x) = (x^{2a+b+c}, e)$$

(b): Let $x = (x^a, e); y = (x^b, e); z = (x^c, p^\gamma)$

$$(x^\lambda \backslash y) \cdot zx = (x^{2a+b+c}, p^\gamma)$$

$$x(yz \cdot x) = (x^{2a+b+c}, p^\gamma)$$

(c): Let $x = (x^a, e); y = (x^b, p^\beta); z = (x^c, e)$

$$(x^\lambda \backslash y) \cdot zx = (x^{2a+b+c}, p^\beta) \ b = even$$

$$x(yz \cdot x) = (x^{2a+b+c}, p^\beta) \ b = even$$

$$(x^\lambda \backslash y) \cdot zx = (x^{2a+3b+c}, p^\beta) \ b = odd$$

$$x(yz \cdot x) = (x^{2a+3b+c}, p^\beta) \ b = odd$$

(d): Let $x = (x^a, p^\alpha); y = (x^b, e); z = (x^c, e)$

$$(x^\lambda \backslash y) \cdot zx = (x^{2a+b+c}, p^{2\alpha}) \ a = even$$
$$x(yz \cdot x) = (x^{2a+b+c}, p^{2\alpha}) \ a = even$$

$$(x^\lambda \backslash y) \cdot zx = (x^{4a+b+c}, p^{2\alpha}) \ a = odd$$
$$x(yz \cdot x) = (x^{4a+b+c}, p^{2\alpha}) \ a = odd$$

(e): Let $x = (x^a, e); y = (x^b, p^\beta); z = (x^c, p^\gamma)$

$$(x^\lambda \backslash y) \cdot zx = (x^{2a+b+c}, p^{\beta+\gamma}) \ b = even$$
$$x(yz \cdot x) = (x^{2a+b+c}, p^{\beta+\gamma}) \ b = even$$

$$(x^\lambda \backslash y) \cdot zx = (x^{2a+3b+c}, p^{\beta+\gamma}) \ b = odd$$
$$x(yz \cdot x) = (x^{2a+3b+c}, p^{\beta+\gamma}) \ b = odd$$

(f): Let $x = (x^a, p^\alpha); y = (x^b, e); z = (x^c, p^\gamma)$

$$(x^\lambda \backslash y) \cdot zx = (x^{2a+b+c}, p^{2\alpha+\gamma}) \ a = even$$
$$x(yz \cdot x) = (x^{2a+b+c}, p^{2\alpha+\gamma}) \ a = even$$

$$(x^\lambda \backslash y) \cdot zx = (x^{4a+b+c}, p^{2\alpha+\gamma}) \ a = odd$$
$$x(yz \cdot x) = (x^{4a+b+c}, p^{2\alpha+\gamma}) \ a = odd$$

(g): Let $x = (x^a, p^\alpha); y = (x^b, p^\beta); z = (x^c, e)$

$$(x^\lambda \backslash y) \cdot zx = (x^{2a+b+c}, p^{2\alpha+\beta}) \ a = even, b = even$$
$$x(yz \cdot x) = (x^{2a+b+c}, p^{2\alpha+\beta}) \ a = even, b = even$$

$$(x^\lambda \backslash y) \cdot zx = (x^{4a+b+c}, p^{2\alpha+\beta}) \ a = odd, b = even$$
$$x(yz \cdot x) = (x^{4a+b+c}, p^{2\alpha+\beta}) \ a = odd, b = even$$

$$(x^\lambda \backslash y) \cdot zx = (x^{2a+3b+c}, p^{2\alpha+\beta}) \ a = even, b = odd$$
$$x(yz \cdot x) = (x^{2a+3b+c}, p^{2\alpha+\beta}) \ a = even, b = odd$$

$$(x^\lambda \backslash y) \cdot zx = (x^{4a+3b+c}, p^{2\alpha+3\beta}) \ a = odd, b = odd$$
$$x(yz \cdot x) = (x^{4a+3b+c}, p^{2\alpha+3\beta}) \ a = odd, b = odd$$

(h): Let $x = (x^a, p^\alpha); y = (x^b, p^\beta); z = (x^c, p^\gamma)$

$$(x^\lambda \backslash y) \cdot zx = (x^{2a+b+c}, p^{2\alpha+\beta+\gamma}) \ a = even, b = even$$
$$x(yz \cdot x) = (x^{2a+b+c}, p^{2\alpha+\beta+\gamma}) \ a = even, b = even$$

$$(x^\lambda \backslash y) \cdot zx = (x^{4a+b+c}, p^{2\alpha+\beta+\gamma}) \ a = odd, b = even$$
$$x(yz \cdot x) = (x^{4a+b+c}, p^{2\alpha+\beta+\gamma}) \ a = odd, b = even$$

$$(x^\lambda \backslash y) \cdot zx = (x^{2a+3b+c}, p^{2\alpha+\beta+\gamma}) \ a = even, b = odd$$
$$x(yz \cdot x) = (x^{2a+3b+c}, p^{2\alpha+\beta+\gamma}) \ a = even, b = odd$$

$$(x^\lambda \backslash y) \cdot zx = (x^{4a+3b+c}, p^{2\alpha+3\beta+\gamma}) \ a = odd, b = odd$$
$$x(yz \cdot x) = (x^{4a+3b+c}, p^{2\alpha+3\beta+\gamma}) \ a = odd, b = odd$$

Since $(x^\lambda \backslash y) \cdot zx = x(yz \cdot x)$ is equal in all the 36 cases considered i.e. whenever $37 \equiv 1(mod\ 2n)$, that is $n = 2, 3, 4, 6, 9, 12, 18$. Also (e, e) is the two sided identity. Moreover, if: $x = (x^a, e)$, then $x^{-1} = (x^{-a}, e)$. If $x = (x^a, p^\alpha)$, then $x^{-1} = (x^{-a}, p^{-a})$ if a is even, and $x^{-1} = (x^{-3a}, p^{-a})$ if a is odd. Therefore, the inverses are defined.

Also for non-associativity:

Let $x = (x^a, p^\alpha); y = (x^b, p^\beta); z = (x^c, p^\gamma)$ where b is an odd integer, then

$$xy \cdot z = (x^{a+3b+c}, p^{\alpha+\beta+\gamma})$$

and

$$x \cdot yz = (x^{a+b+c}, p^{\alpha+\beta+\gamma})$$

therefore,$xy \cdot z \neq x \cdot yz$. Thus the construction is non-associative except when $n = 2$ which gives the group $C_4 \times C_2$. The construction exists as Moufang loop when $n = 3$ (compare Theorem 2.1). Hence, it is an Osborn loop of order $4n$, n = 4,6,9,12 and 18.

Next, we examine the universality of the constructed Osborn loop using [5] as done in [6].

An Osborn loop $I(\cdot) = C_{2n} \times C_2$ is universal if $I(\cdot)$ obeys:

$$y \cdot yy = y^\lambda \backslash y \cdot y$$

Let $y = (x^b, p^\beta)$, where b is an odd integer.

$$y \cdot yy = (x^{13b}, p^{3\beta}) \ b = odd$$
$$y^\lambda \backslash y \cdot y = (x^{7b}, p^{3\beta}) \ b = odd$$

Thus,

$$y \cdot yy \neq y^\lambda \backslash y \cdot y$$

Hence, $I(\cdot) = C_{2n} \times C_2$ as defined in construction 1 is not a universal Osborn loop.

Example 3. *Construction* 2
Let $I(\cdot) = C_{2n} \times C_2$ *that is* $I = \{(x^\alpha, p^\beta), 0 \leq \alpha \leq 2n-1, 0 \leq \beta \leq 1\}$ *and the binary operation is defined as follows:*

$$(x^a, e) \cdot (x^b, p^\beta) = (x^{a+b}, p^\beta)$$
$$(x^a, p^\alpha) \cdot (x^b, e) = (x^{a+b}, p^\alpha)$$
$$(x^a, p^\alpha) \cdot (x^b, p^\beta) = (x^{a+b}, p^{\alpha+\beta}) \ if \ b \equiv 0(mod \ 2)$$
$$= (x^{a-b}, p^{\alpha+\beta}) \ if \ b \equiv 1(mod \ 2)$$
$$(x^a, p^\alpha) \cdot (x^b, p^\beta) = (x^{a-b}, p^{\alpha-\beta}) \ if \ a \equiv 1(mod \ 2), b \equiv 1(mod \ 2)$$
$$(x^{b+c}, p^\delta) \cdot (x^a, p^\alpha) = (x^{a+b+c}, p^{\alpha+\delta}) \ if \ b \equiv 0(mod \ 2)$$
$$(x^{b+c}, p^\delta) \cdot (x^a, p^\alpha) = (x^{a-b+c}, p^{\alpha+\delta}) \ if \ b \equiv 1(mod \ 2)$$
$$(x^{b+c}, p^{\beta+\gamma}) \cdot (x^a, p^\alpha) = (x^{c-a-b}, p^{\alpha-\beta+\gamma}) \ if \ a \equiv 1(mod \ 2)$$
$$, b \equiv 1(mod \ 2)$$
$$(x^{b+c}, p^{\beta+\gamma}) \cdot (x^a, p^\alpha) = (x^{b+c-a}, p^{\beta+\gamma-\alpha}) \ if \ a \equiv 1(mod \ 2)$$

Then $I(\cdot)$ *is an Osborn loop of order* $4n$, *where* $n = 2$, 3, 4, 6, 9, 12 *and* 18.

Proof

We first show that $I(\cdot)$ satisfies Osborn identity (1):
Then we follow up as in construction 1, and establish that construction 2 is non-associative and non-universal Osborn loop.

Example 4. *Construction* 3
Let $I(\cdot) = C_{2n} \times C_2$ *that is* $I = \{(x^\alpha, p^\beta), 0 \leq \alpha \leq 2n-1, 0 \leq \beta \leq 1\}$ *and the binary operation is defined as follows:*

$$(x^a, e) \cdot (x^b, p^\beta) = (x^{a+b}, p^\beta)$$
$$(x^a, p^\alpha) \cdot (x^b, e) = (x^{a+b}, p^\alpha)$$
$$(x^a, p^\alpha) \cdot (x^b, p^\beta) = (x^{a+b}, p^{\alpha+\beta}) \ if \ b \equiv 0(mod \ 2)$$
$$= (x^a, p^{\alpha+\beta}) \ if \ b \equiv 1(mod \ 2)$$
$$(x^a, p^\alpha) \cdot (x^b, p^\beta) = (x^a, p^\alpha) \ if \ a \equiv 1(mod \ 2), b \equiv 1(mod \ 2)$$
$$(x^{b+c}, p^\delta) \cdot (x^a, p^\alpha) = (x^{a+b+c}, p^{\alpha+\delta}) \ if \ b \equiv 0(mod \ 2)$$
$$(x^{b+c}, p^\delta) \cdot (x^a, p^\alpha) = (x^{a+c}, p^{\alpha+\delta}) \ if \ b \equiv 1(mod \ 2)$$
$$(x^{b+c}, p^{\beta+\gamma}) \cdot (x^a, p^\alpha) = (x^c, p^{\alpha+\gamma}) \ if \ a \equiv 1(mod \ 2), b \equiv 1(mod \ 2)$$
$$(x^{b+c}, p^{\beta+\gamma}) \cdot (x^a, p^\alpha) = (x^{b+c}, p^{\beta+\gamma}) \ if \ a \equiv 1(mod \ 2)$$

Then $I(\cdot)$ *is an Osborn loop of order* $4n$, *where* $n = 2$, 3, 4, 6, 9, 12 *and* 18

Proof
We first show that $I(\cdot)$ satisfies Osborn identity (1):
Then we follow up as in construction 1, and establish that construction 3 is non-associative and non-universal Osborn loop.

Example 5. *Construction* 4
Let $I(\cdot) = C_{2n} \times C_2$ *that is* $I = \{(x^{\alpha}, p^{\beta}), 0 \le \alpha \le 2n-1, 0 \le \beta \le 1\}$ *and the binary operation is defined as follows:*

$$(x^a, e) \cdot (x^b, p^{\beta}) = (x^{a+b}, p^{\beta})$$
$$(x^a, p^{\alpha}) \cdot (x^b, e) = (x^{a+b}, p^{\alpha})$$
$$(x^a, p^{\alpha}) \cdot (x^b, p^{\beta}) = (x^{a+b}, p^{\alpha+\beta}) \ if \ b \equiv 0(mod\ 2)$$
$$= (x^{a+kb}, p^{\alpha+\beta}) \ if \ a \equiv 0(mod\ 2), b \equiv 1(mod\ 2)$$
$$(x^a, p^{\alpha}) \cdot (x^b, p^{\beta}) = (x^{a+kb}, p^{\alpha+k\beta}) \ if \ a \equiv 1(mod\ 2), b \equiv 1(mod\ 2)$$
$$(x^{b+c}, p^{\delta}) \cdot (x^a, p^{\alpha}) = (x^{a+b+c}, p^{\alpha+\delta}) \ if \ a \equiv 0(mod\ 2), b \equiv 0(mod\ 2)$$
$$(x^{b+c}, p^{\delta}) \cdot (x^a, p^{\alpha}) = (x^{a+kb+c}, p^{\alpha+\delta}) \ if \ a \equiv 0(mod\ 2), b \equiv 1(mod\ 2)$$
$$(x^{b+c}, p^{\beta+\gamma}) \cdot (x^a, p^{\alpha}) = (x^{b+c+ka}, p^{\beta+\gamma+k\alpha}) \ if \ a \equiv 1(mod\ 2)$$
$$, b \equiv 0(mod\ 2)$$
$$(x^{b+c}, p^{\beta+\gamma}) \cdot (x^a, p^{\alpha}) = (x^{c+ka+kb}, p^{\alpha+k\beta+\gamma}) \ if \ a \equiv 1(mod\ 2)$$
$$, b \equiv 1(mod\ 2)$$

Then $I(\cdot)$ *is an Osborn loop of order* $4n$, *where* n = 2, 3, 4, 6, 9, 12 *and* 18

Proof
We first show that $I(\cdot)$ satisfies Osborn identity (1):
Then we follow up as in construction $1-3$ above, and establish that construction 4 is non-associative and non-universal Osborn loop except when $k = 1$.

4. SOME PROPERTIES OF NON-UNIVERSAL OSBORN LOOPS

Theorem 3. *A non-universal Osborn loop is not a CIP loop*

Proof
Let $x = (x^a, p^{\alpha}), y = (x^b, p^{\beta})$ b=odd. By direct computation we have

$$xy \cdot x^{\rho} = (x^{kb}, p^{\beta}) \neq y$$

Again if $k = 1$, we have $xy \cdot x^{\rho} = (x^b, p^{\beta}) = y$
Also, let $x = (x^a, p^{\alpha}), y = (x^b, p^{\beta})$ a=odd
Then $xy \cdot x^{\rho} = (x^{a(1-k)+b}, p^{\beta}) \neq y$ except when $k = 1$. Thus a non-universal Osborn loop is not a CIP loop.

Theorem 4. *A non-unversal Osborn loop is not a WIPL.*

Proof
Let $x = (x^a, p^\alpha), y = (x^b, p^\beta)$ a=odd
then

$$x(yx)^\rho = (x^{a(1-k)-b}, p^{-\beta})$$

If $k = 1$, we have $x(yx)^\rho = (x^{-b}, p^{-\beta})$. i.e. the constructions are not a WIPL except at $k = 1$.

Theorem 5. *A non-universal Osborn loop is not an AIPL.*

Proof
Let $x = (x^a, p^\alpha), y = (x^b, p^\beta)$ b=odd
then

$$(xy)^\rho = (x^{-(a+kb)}, p^{(-\alpha+\beta)})$$

and

$$x^\rho y^\rho = (x^{-(a+k^2b)}, p^{-(\alpha+\beta)})$$

$$(xy)^\rho = (x^{-(a+kb)}, p^{(-\alpha+\beta)}) \neq (x^{-(a+k^2b)}, p^{-(\alpha+\beta)})$$

Hence $(xy)^\rho \neq x^\rho y^\rho$ except when $k = 1$.

Theorem 6. *A non-universal Osborn loop is not an AAIP loop*

Proof
Let $x = (x^a, p^\alpha), y = (x^b, p^\beta)$ a=odd, b=odd
then

$$(xy)^\rho = (x^{-(a+kb)}, p^{-(\alpha+k\beta)})$$

and

$$y^\rho x^\rho = (x^{-(kb+k^2a)}, p^{-(k\alpha+\beta)})$$

Hence $(xy)^\rho \neq y^\rho x^\rho$ except when $k = 1$
Or

$$(xy)^\lambda = (x^{-(a+kb)}, p^{-(\alpha+k\beta)}) \; a = odd, b = odd$$

$$y^\lambda x^\lambda = (x^{-(kb+k^2a)}, p^{-(k\alpha+\beta)}) \; a = odd, b = odd$$

Hence $(xy)^\lambda \neq y^\lambda x^\lambda$ except when $k = 1$

Theorem 7. *A non-universal Osborn loop is not a LIP or RIP loop*

Proof Let $y = (x^b, p^\beta), x = (x^a, p^\alpha), x^\lambda = (x^{-a}, p^{-\alpha})$ a=even, b=odd

$$x^\lambda \cdot xy \neq y$$

except when $k = 1$
Next,
Let $y = (x^b, p^\beta), x = (x^a, p^\alpha), x^\rho = (x^{-ka}, p^{-\alpha})$ a=odd, b=even

$$yx \cdot x^\rho = (x^{k(a-ka)+b}, p^\beta)$$

Thus $yx \cdot x^\rho \neq y$ except when $k = 1$. The proof is complete.

Theorem 8. *A non-universal Osborn loop is not a LAP Or RAP loop*

Proof
Let $y = (x^b, p^\beta), x = (x^a, p^\alpha)$

$$xx \cdot y = (x^{a+b+ka}, p^{2\alpha+\beta}) \ a = odd$$

Now,

$$x \cdot xy = (x^{2a+b}, p^{2\alpha+\beta}) \ a = odd$$

Thus $xx \cdot y \neq x \cdot xy$ except when $k = 1$

Next, consider:

$$y \cdot xx = yx \cdot x \ (RAP)$$

Let $x = (x^a, p^\alpha), y = (x^b, p^\beta)$ a=odd

$$y \cdot xx = (x^{a+b+ka}, p^{2\alpha+\beta})$$

and,

$$yx \cdot x = (x^{2ka+b}, p^{2\alpha+\beta})$$

Thus $y \cdot xx \neq yx \cdot x$ except when $k = 1$.

Theorem 9. *A non-universal Osborn is not flexible, power associative and diassociative loop.*

Proof
Let $x = (x^a, p^\alpha), y = (x^b, p^\beta)$

$$xy \cdot x = (x^{2a+kb}, p^{2\alpha+\beta})$$

$$x \cdot yx = (x^{2a+b}, p^{2\alpha+\beta})$$

Thus $xx \cdot x \neq x \cdot xx$ except when $k = 1$ Next, we look at power associativity: Let $x = (x^a, p^\alpha)$ a=odd

$$xx \cdot x = (x^{a+2ka}, p^{3\alpha})$$

$$x \cdot xx = (x^{a+(k^2+k)a}, p^{3\alpha})$$

Thus $xx \cdot x \neq x \cdot xx$ except when $k = 1$. Finally, we consider diassociativity:

$$x \cdot yy = (x^{a+(k+k^2)b}, p^{\alpha+(1+k)\beta})\ a = odd$$

$$xy \cdot y = (x^{a+2kb}, p^{\alpha+2\beta})\ a = odd$$

Thus $x \cdot yy \neq xy \cdot y$ except when $k = 1$.

Theorem 10. *A non-universal Osborn loop is not a SAIP loop.*

Proof:
Let $x = (x^a, p^\alpha), y = (x^b, p^\beta)$

$$(xy \cdot x)^\rho = (x^{-(2a+kb)}, p^{-(2\alpha+\beta)})\ b = odd$$

and

$$x^\rho y^\rho \cdot x^\rho = (x^{-(2a+k^2b)}, p^{-(2\alpha+\beta)})\ b = odd$$

Thus $(xy \cdot x)^\rho \neq x^\rho y^\rho \cdot x^\rho$ except when $k = 1$

Theorem 11. *A non-universal Osborn loop is not a CC-loop.*

Proof:
Let $x = (x^a, p^\alpha), y = (x^b, p^\beta), z = (x^c, p^\gamma)$

$$x \cdot yz = (x^{a+b+c}, p^{\alpha+\beta+\gamma})\ b = odd$$

$$(xy)/x \cdot xz = (x^{a+kb+c}, p^{\alpha+\beta+\gamma})\ b = odd$$

Thus $x \cdot yz \neq (xy)/x \cdot xz$ except when $k = 1$.
Next, consider:

$$zy \cdot x = (x^{ka+b+c}, p^{\alpha+\beta+\gamma})\ a = odd$$

$$zx \cdot x\backslash(yx) = (x^{2ka+b+c-a}, p^{\alpha+\beta+\gamma})\ a = odd$$

Thus $zy \cdot x \neq zx \cdot x\backslash(yx)$. This completes the proof.

Corollary 1. *A non-universal Osborn loop is not a G-loop*

Proof:
The necessary and sufficient condition for a loop to be a G-loop is that it is a CC-loop [2]. Then from theorem 4.9, the proof follows.

Theorem 12. *For a finite non-universal Osborn loop the nuclei do not coincide.*

Proof:
Let $x = (x^a, p^\alpha), y = (x^b, p^\beta), z = (x^c, p^\gamma)$

$$zx \cdot y = (x^{a+kb+c}, p^{\alpha+\beta+\gamma})$$

Therefore, $N_\lambda(I,\cdot) = (x^{a+kb+c}, p^{\alpha+\beta+\gamma})$ b=odd
Next, consider:

$$y \cdot xz = (x^{a+b+c}, p^{\alpha+\beta+\gamma}) \; b = odd$$

Therefore,

$$N_\rho(I,\cdot) = (x^{a+b+c}, p^{\alpha+\beta+\gamma}) \; b = odd$$

Also,

$$yz \cdot x = (x^{ka+b+c}, p^{\alpha+\beta+\gamma}) \; a = odd$$
$$y \cdot zx = (x^{ka+b+c}, p^{\alpha+\beta+\gamma}) \; a = odd$$

Thus,

$$N_\mu(I,\cdot) = (x^{ka+b+c}, p^{\alpha+\beta+\gamma}) \; a = odd$$

Hence,

$$N(I,\cdot) = N_\lambda(I,\cdot) \cap N_\rho(I,\cdot) \cap N_\mu(I,\cdot)$$

Therefore,

$$N(I,\cdot) = (x^{a+b+c}, p^{\alpha+\beta+\gamma}) \; a = odd, b = odd$$

Thus,

$$N(I,\cdot) = N_\rho(I,\cdot) \neq N_\lambda(I,\cdot) \neq N_\mu(I,\cdot)$$

Remark 1. *For a WIPL, the four nuclei $N, N_\lambda, N_\rho, N_\mu$. coincide, i.e. $N = N_\lambda = N_\rho = N_\mu$. The same is true for a CIPL and an IPL since they are WIPLs as observed by Osborn himself* [9]. *Since, the constructions above are not WIPL, CIPL nor IPL, the nuclei are not equal.*

Lemma 3. *For a finite non-universal Osborn loop, elements in the centrium and in the center whenever they exist are equal*

Proof:
Let $x = (x^a, p^\alpha), y = (x^b, p^\beta), z = (x^c, p^\gamma)$

$$C(I,\cdot) = (x^{a+c}, p^{\alpha+\gamma}) \; a = even$$

Or

$$C(I,\cdot) = (x^{b+c}, p^{\beta+\gamma}) \; b = even$$

Next,

$$Z(I,\cdot) = (x^{b+c}, p^{\beta+\gamma}) \; b = even$$

Proposition 1. *An Osborn loop $(G,\cdot)$ is universal if and only if it is a 3-power associative property loop (3-PAPL)*

Proof:
A left universal Osborn loop G obeys the identity: $v \cdot vv = v^{\lambda} \backslash v \cdot v \ \forall \ v \in G$. See detail in [6]. Then $v \cdot vv = v^{\lambda} \backslash v \cdot v = vv \cdot v$
Thus, $v \cdot vv = vv \cdot v$
Conversely,

$$vv \cdot v = vL_v.v = v^{\lambda} \backslash v \cdot v$$

Thus,

$$v^{\lambda} \backslash v \cdot v = v \cdot vv$$

Remark 2. *This is the necessary and sufficient condition for an Osborn loop to be universal.*

Corollary 2. *A moufang loop that is a 3-power associative property loop is universal*

Proof:
Since a moufang loop satisfies the autotopism

$$T = (L_x, R_x, R_x L_x)$$

Then it is a variety of an Osborn loop. It follows from prop.4.1 above.

Theorem 13. *A non-universal Osborn loop is not a generalized Moufang loop.*

Proof:
A loop $I(\cdot)$ is a generalized Moufang loop, if one of the identifies:

$$x(yz \cdot x) = (y^{\lambda} x^{\lambda})^{\rho} \cdot zx$$

or

$$(x \cdot zy)x = xz \cdot (x^{\rho} y^{\rho})^{\lambda}$$

holds in $I(\cdot)$.

$$x(yz \cdot x) = (x^{2a+b+c}, p^{2\alpha+\beta+\gamma}) \ a = even$$
$$(y^{\lambda} x^{\lambda})^{\rho} \cdot zx = (x^{2a+b+c}, p^{2\alpha+\beta+\gamma}) \ a = even$$
$$(x \cdot zy)x = (x^{a+ka+b+c}, p^{2\alpha+\beta+\gamma}) \ a = odd$$
$$xz \cdot (x^{\rho} y^{\rho})^{\lambda} = (x^{a+k(a+b)+c}, p^{2\alpha+\beta+\gamma}) \ a = odd$$

The proof is complete.

Remark 3. *Whenever* $k = 1$ *and* $n = 3$ *(atmost) where the example exists as a group and as a moufang loop respectively, then it is a generalized Moufang*

Remark 4. *This section 4 is simply a characterization of non-universal Osborn loops.*

5. CONCLUDING REMARKS

The Examples above are examples of finite non-universal Osborn loops. The constructions are new methods of constructing Osborn loops that are not universal. This becomes pertinent because most popular varieties of Osborn loops, like VD loops, CC-loops, Moufang loops and universal WIPLs are universal. It becomes interesting to researchers as to whether there are other varieties of Osborn loops that are not universal. This work would be giving Osborn loops a new dimension. This new branch, the non-universal Osborn loops is yet to gain a popular status comparatively.

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